# Fused Angles for Body Orientation Representation

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*Abstract*— **The parameterisation of rotations in three dimensional Euclidean space is an area of applied mathematics that has long been studied, dating back to the original works of Leonhard Euler in the 18th century. As such, many ways of parameterising a rotation have been developed over the years. Motivated by the task of representing the orientation of a balancing body, the fused angles parameterisation is developed and introduced in this paper. This novel representation is carefully defined both mathematically and geometrically, and thoroughly investigated in terms of the properties it possesses, and how it relates to other existing representations. A second intermediate representation, tilt angles, is also introduced as a natural consequence thereof.**

## I. Introduction

Numerous ways of representing a rotation in three-dimensional Euclidean space have been developed and refined over the years. Many of these representations, also referred to as parameterisations, arose naturally from classical mathematics and have found widespread use in areas such as physics, engineering and robotics. Prominent examples of such representations include rotation matrices, quaternions and Euler angles. In this paper, a new parameterisation of the manifold of all three-dimensional rotations is proposed. This parameterisation, referred to as *fused angles*, was motivated by the analysis and control of the balance of bodies in 3D and the shortcomings of the various existing rotation representations to describe the state of balance in an intuitive and problem-relevant way. More specifically, the advent of fused angles was to address the problem of representing the orientation of a body in an environment where there is a clear notion of what is 'up', defined implicitly, for example, through the presence of gravity. An orientation is just a rotation relative to some global fixed frame however, so fused angles can equally be used to represent any arbitrary three-dimensional rotation, much like Euler angles can be used for both purposes for example. The shortcomings of Euler angles that make them unsuitable for this balance-inspired task are discussed in detail in Section II-D.

When analysing the balance state of a body, such as for example a humanoid robot, it is very helpful to be able to work with a parameterisation of the orientation that yields information about the components of the rotation within each of the three major planes, i.e. within the $\mathbf{yz}$, $\mathbf{xz}$ and $\mathbf{xy}$ planes. These components of the rotation can conceptually be thought of as a way of quantifying the 'amount of rotation' about the x, y and z-axes respectively. It is desirable for these components to each offer a useful geometric interpretation, and behave intuitively throughout the rotation space, most critically not sacrificing axisymmetry within the horizontal $\mathbf{xy}$ plane by the introduction of a clear sequential order of rotations. The fusing of individual rotation components to avoid such an order motivated the term 'fused angles'. Quaternions, a common choice of parameterisation in computational environments, clearly do not address these requirements, as elucidated in Section II-C.

The fused angles rotation representation has to date found a number of uses. Most recently in work published by the same authors, an attitude estimator was formulated that internally relied on the concept of fused angles [1] [2]. The open source ROS software for the NimbRo-OP humanoid robot [3], developed by the University of Bonn, also relies on the use of fused angles, most notably in the areas of state estimation and walking [4] [5]. Furthermore, a Matlab and Octave library [6] targeted at the numerical and computational handling of all manners of three-dimensional rotation representations, including fused angles, has been released. This library is intended to serve as a common reference for the implementation in other programming languages of a wide range of conversion and computation functions. It is seen by the authors as a test bed to support the development of new rotation-related algorithms.

The convention is used in the following work that the global z-axis points in the 'up' direction relative to the environment. As mentioned previously, this accepted 'up' direction will almost always be defined as the antipodal direction of gravity. This ensures that definitions such as that of *fused yaw* make terminological sense in consideration of the true rotation of a body relative to its environment. All derived formulas and results could easily be rewritten using an alternative convention if this were to be desired.

The contribution of this paper lies in the introduction of the novel concept of fused angles for the representation of rotations. A further contribution is the concept of tilt angles (see Section III), an intermediary representation that emerges naturally from the derivation of the former.

## II. Review of Existing Rotation Representations

Many ways of representing 3D rotations in terms of a finite set of parameters exist. Different representations have different advantages and disadvantages, and which representation is suitable for a particular application depends on a wide range of considerations. Such considerations include:

- Ease of geometric interpretation, in particular in a form that is relevant to the particular problem,
- The range of singularity-free behaviour,

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- Computational efficiency in terms of common operations such as rotation composition and vector rotation,
- Mathematical convenience, in terms of numeric and algebraic complexity and manipulability, and
- Algorithmic convenience, in the sense of a representation potentially possessing properties that can conveniently be exploited for a particular algorithm.

A wide range of existing rotation representations are reviewed in this section as a basis for comparison. Due to the dimensionality of the space of 3D rotations, a minimum of three parameters are required for any such representation. A representation with exactly three parameters is referred to as *minimal*, while other representations with a greater number of parameters are referred to as *redundant*.

### A. Rotation Matrices

A rotation can be represented as a linear transformation of coordinate frame basis vectors, expressed in the form of an orthogonal matrix of unit determinant. Due to the strong link between such transformation matrices and the theory of direction cosines, the name Direction Cosine Matrix is sometimes used. The space of all rotation matrices is called the special orthogonal group $\mathrm{SO}(3)$, and is defined as

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^T R = \mathbb{I},\, \det(R) = 1\}. \tag{1}$$

Rotation of a vector $\mathbf{v} \in \mathbb{R}^3$ by a rotation matrix is given by matrix multiplication. For a rotation from coordinate frame {G} to {B}, we have that

$${}^G_B R = \begin{bmatrix} {}^G\mathbf{x}_B & {}^G\mathbf{y}_B & {}^G\mathbf{z}_B \end{bmatrix} = \begin{bmatrix} {}^B\mathbf{x}_G & {}^B\mathbf{y}_G & {}^B\mathbf{z}_G \end{bmatrix}^T, \tag{2}$$

where ${}^G\mathbf{y}_B$, for example, is the column vector corresponding to the y-axis of frame {B}, expressed in the coordinates of frame {G}. The notation ${}^G_B R$ refers to the relative rotation from {G} to {B}. With nine parameters, rotation matrices are clearly a redundant parameterisation of the rotation space. They are quite useful in that they are free of singularities and trivially expose the basis vectors of the fixed and rotated frames, but for many tasks they are not as computationally and numerically suitable as other representations.

### B. Axis-Angle and Rotation Vector Representations

As stated by Euler's rotation theorem, every rotation in the three-dimensional Euclidean space $\mathbb{R}^3$ can be expressed as a single rotation about some axis. As such, each rotation can be mapped to a pair $(\hat{\mathbf{u}}, \theta) \in S^2 \times \mathbb{R}$, where $\hat{\mathbf{u}}$ is a unit vector corresponding to the axis of rotation, and $\theta$ is the magnitude of the rotation. Note that $S^2 = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| = 1\}$, the 2-sphere, is the set of all unit vectors in $\mathbb{R}^3$. A closely related concept is that of the rotation vector, given by $\mathbf{u} = \theta\hat{\mathbf{u}}$, which encodes the angle of rotation as the magnitude of the vector defining the rotation axis. Both the axis-angle and rotation vector representations suffer from a general impracticality of mathematical and numerical manipulation. For example, no formula for rotation composition exists that is more direct than converting to quaternions and back. The Simultaneous Orthogonal Rotations Angle (SORA) vector, a slight reformulation of the rotation vector concept in terms of virtual angular velocities and virtual time, was presented by Sašo Tomažič et al. in [7]. This formulation suffers from drawbacks similar to those of the rotation vector representation. These drawbacks include a discontinuity at rotations of $180^\circ$, and a general lack of geometric intuitiveness.

### C. Quaternions

The set of all quaternions $\mathbb{H}$, and the subset $\mathbb{Q}$ thereof of all quaternions that represent pure rotations, are defined as

$$\begin{aligned} \mathbb{H} &= \{q = (q_0, \mathbf{q}) \equiv (w, x, y, z) \in \mathbb{R}^4\}, \\ \mathbb{Q} &= \{q \in \mathbb{H} : |q| = 1\}. \end{aligned} \tag{3}$$

Quaternion rotations can be related to the axis-angle representation, and thereby visualised to some degree, using

$$q = (q_0, \mathbf{q}) = \left(\cos\tfrac{\theta}{2}, \hat{\mathbf{u}}\sin\tfrac{\theta}{2}\right) \in \mathbb{Q}, \tag{4}$$

where $(\hat{\mathbf{u}}, \theta) \in S^2 \times \mathbb{R}$ is any axis-angle rotation pair, and $q$ is the equivalent quaternion rotation. The use of quaternions to express rotations generally allows for very computationally efficient calculations, and is grounded by the well-established field of quaternion mathematics. A crucial advantage of the quaternion representation is that it is free of singularities. On the other hand however, it is not a one-to-one mapping of the special orthogonal group, as $q$ and $-q$ both correspond to the same rotation. The redundancy of the parameters also means that the unit magnitude constraint has to explicitly and sometimes non-trivially be enforced in numerical computations. Furthermore, no clear geometric interpretation of quaternions exists beyond the implicit relation to the axis-angle representation given in (4). For applications related to the balance of a body, where questions arise such as 'how rotated' a body is in total or within a particular major plane, the quaternion representation yields no direct insight.

### D. Euler Angles

A step in the right direction of understanding the different components of a rotation is the notion of Euler angles. In this representation, the total rotation is split into three individual elemental rotations, each about a particular coordinate frame axis. The three Euler angles $(\alpha, \beta, \gamma)$ describing a rotation are the successive magnitudes of these three elemental rotations. Many conventions of Euler angles exist, depending on the order in which the elemental axis rotations are chosen and whether the elemental rotations are taken to be intrinsic (about the rotating coordinate frame) or extrinsic (about the fixed coordinate frame). Extrinsic Euler angles can easily be mapped to their equivalent intrinsic Euler angles representations, and so the two types do not exhibit fundamentally different behaviour. If all three coordinate axes are used in the elemental rotations, the representation is alternatively known as Tait-Bryan angles, and the three parameters are referred to as yaw, pitch and roll, respectively. Tait-Bryan angles, although promising on first thought, do not suffice for the representation of the orientation of a body in balance-related scenarios. The main reasons for this are:

- The proximity of the gimbal lock singularity to normal working ranges, leading to unwanted artefacts due to increased local parameter sensitivity in a widened neighbourhood of the singularity,

- The fundamental requirement of an order of elemental rotations, leading to asymmetrical definitions of pitch and roll that do not mirror one another in behaviour,
- The asymmetry introduced by the use of a yaw definition that depends on the projection of one of the coordinate axes onto a fixed plane, leading to unintuitive non-axisymmetric behaviour of the yaw angle.

As an example of the last of these points, consider the intrinsic ZYX Euler angles representation and the previously discussed convention that the global z-axis points ‘upwards’ (see Section I). Consider a body in space, assumed to be in its identity orientation, and some arbitrary rotation of the body relative to its environment. It would be natural and intuitive to expect that the yaw of this rotation is independent of the chosen definition of the global x and y-axes. This is because the true rotation of the body is always the same, regardless of the essentially arbitrary choice of the global x and y-axes, and one would expect a well-defined yaw to be a property of the rotation, not the axis convention. This is not the case for ZYX Euler yaw however, as can be verified by counterexample with virtually any non-degenerate case. The yaw component of the fused angles representation, defined in Section III, can be proven to satisfy this property.

### E. Vectorial Parameterisations

Parameterisations are sometimes developed specifically to exhibit certain properties that can be exploited to increase the efficiency of an algorithm. A class of such generally more mathematical and abstract rotation representations is given by the family of vectorial parameterisations. Named examples of these include the Gibbs-Rodrigues parameters and the Wiener-Milenković parameters, also known as the conformal rotation vector (CRV). Such parameterisations derive from mathematical identities such as the Euler-Rodrigues formula, and as such do not in general have any useful geometric interpretation, and find practical use in only very specific applications. Detailed derivations and analyses of vectorial parameterisations can be found in [8] and [9].

## III. Fused Angles

The idea of fused angles was motivated by the lack of an existing 3D rotation formalism that naturally deals with the dissolution of a complete rotation into parameters that are specifically and geometrically relevant to the balance of a body, and does not introduce order-based asymmetry in the parameters. None of the representations discussed in Section II satisfy this property. The unwanted artefacts in the existing notions of yaw (see Section II-D) also led to the need for a more suitable, stable and axisymmetric definition of yaw. This section defines the fused angles representation and motivates the underlying mathematical derivation thereof.

We begin by defining an intermediate rotation representation, referred to as *tilt angles*. The tilt angles parameter definitions are shown graphically in Fig. 1. Let {G} denote the global fixed frame, defined with the convention that the global z-axis points upwards in the environment, as discussed in Section I. We define {B} to be the body-fixed coordinate frame. For an identity orientation of the body, the frames {G} and {B} should clearly coincide. Throughout this paper we use the notation that ${}^G\mathbf{z}_B = \big({}^G z_{Bx}, {}^G z_{By}, {}^G z_{Bz}\big)$, for example, denotes the unit vector corresponding to the positive z-axis of frame {B}, expressed in the coordinates of frame {G}. The absence of a coordinate basis qualifier implies that a vector is expressed relative to {G}.

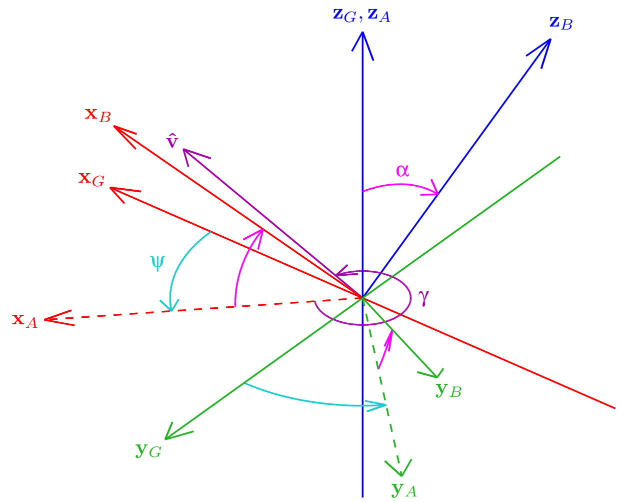


Fig. 1. Definition of the tilt rotation and tilt angles parameters $(\psi, \gamma, \alpha)$.

As $\mathbf{z}_G$ and $\mathbf{z}_B$ are vectors in $\mathbb{R}^3$, a rotation about an axis perpendicular to both vectors exists that maps $\mathbf{z}_G$ onto $\mathbf{z}_B$. We choose an axis-angle representation $(\hat{\mathbf{v}}, \alpha)$ of this *tilt rotation* such that $\alpha \in [0, \pi]$. The angle $\alpha$ is referred to as the *tilt angle* of {B}, and the vector $\hat{\mathbf{v}}$ is referred to as the *tilt axis* of {B}. We define coordinate frame {A} to be the frame that results when we apply the inverse of the tilt rotation to {B}. By definition $\mathbf{z}_A = \mathbf{z}_G$, so it follows that $\hat{\mathbf{v}}$—and trivially also $\mathbf{x}_A$—must lie in the $\mathbf{x}_A\mathbf{y}_A$ plane. The angle $\gamma$ about $\mathbf{z}_A$ from $\mathbf{x}_A$ to $\hat{\mathbf{v}}$ is referred to as the *tilt axis angle* of {B}. The tilt rotation from {A} to {B} is completely defined by the parameter pair $(\gamma, \alpha)$. We now note that the rotation from {G} to {A} is one of pure yaw, that is, a pure z-rotation, and so define the angle $\psi$ about $\mathbf{z}_G$ from $\mathbf{x}_G$ to $\mathbf{x}_A$ as the *fused yaw* of {B}. It is important to note that the choice of using the x-axes in this definition of yaw is arbitrary, and a similar definition using the y-axes would be completely equivalent. The complete tilt angles representation of the rotation from {G} to {B} is now defined as

$$ {}^G_B T = (\psi, \gamma, \alpha) \in (-\pi, \pi] \times (-\pi, \pi] \times [0, \pi] \equiv \mathbb{T}. \tag{5}$$

From the method of construction it can be seen that all rotations possess a tilt angles representation, but this is not always unique. Most notably when $\alpha = 0$, the $\gamma$ parameter can be arbitrary. To remedy this, we introduce the concepts of *fused pitch* and *fused roll*. Let $\mathbf{v}_x$ and $\mathbf{v}_y$ be the projections of the $\mathbf{z}_G$ vector onto the body-fixed $\mathbf{y}_B\mathbf{z}_B$ and $\mathbf{x}_B\mathbf{z}_B$ planes respectively. We define the fused pitch of {B} as the angle $\theta$ between $\mathbf{z}_G$ and $\mathbf{v}_x$, of sign $-\operatorname{sgn}\big({}^B\mathbf{z}_{Gx}\big)$. By logical

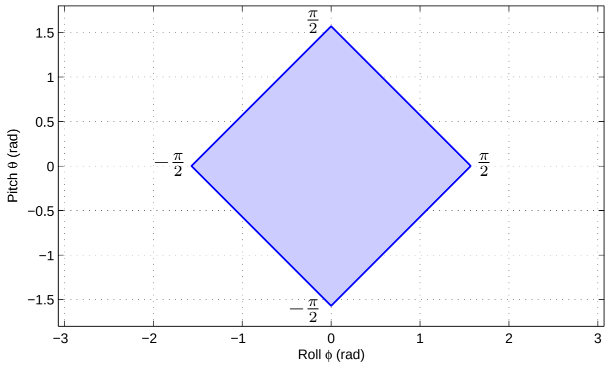


Fig. 2. Valid domain of fused pitch and roll, as per the sine sum criterion.

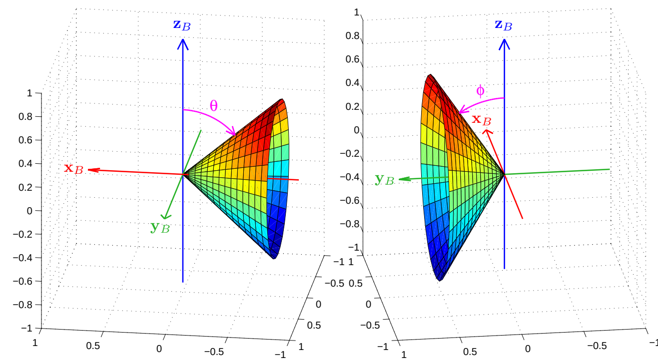

Fig. 3. Cones of constant fused pitch $\theta$ (left) and fused roll $\phi$ (right).

completion, the magnitude of $\theta$ is taken to be $\frac{\pi}{2}$ if $\mathbf{v}_x = \mathbf{0}$. We similarly define the fused roll of $\{\mathrm{B}\}$ as the angle $\phi$ between $\mathbf{z}_G$ and $\mathbf{v}_y$, of sign $\mathrm{sgn}\big({}^B\mathbf{z}_{Gy}\big)$. The magnitude of $\phi$ is taken to be $\frac{\pi}{2}$ if $\mathbf{v}_y = \mathbf{0}$. Conceptually, fused pitch and roll can simply be thought of as the angles between $\mathbf{z}_G$ and the $\mathbf{y}_B\mathbf{z}_B$ and $\mathbf{x}_B\mathbf{z}_B$ planes, respectively.

From inspection it can be seen that the fused pitch and roll only uniquely specify a tilt rotation up to the z-hemisphere. To resolve this ambiguity, the *hemisphere* of a rotation is defined as $\mathrm{sign}\big({}^B\mathbf{z}_{Gz}\big) = \mathrm{sign}\big({}^G\mathbf{z}_{Bz}\big)$, where $\mathrm{sign}(0) = 1$. The complete fused angles representation of the rotation from $\{\mathrm{G}\}$ to $\{\mathrm{B}\}$ can now be defined as

$$\begin{aligned} {}^G_BF &= (\psi, \theta, \phi, h) \\ &\in (-\pi, \pi] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times \{-1, 1\} \equiv \hat{\mathbb{F}}. \end{aligned} \tag{6}$$

The $(\theta, \phi, h)$ triplet in (6) replaces the $(\gamma, \alpha)$ pair from the the tilt angles representation to define the tilt rotation. Although also evident by a geometrical argument, mathematical analysis of the above definitions reveals that the tilt rotation depends only on the value of ${}^B\mathbf{z}_G$, and vice versa. As such, the following expressions can be derived as an alternate mathematical definition of the tilt rotation parameters:

$$\gamma = \mathrm{atan2}\big(-{}^Bz_{Gx}, {}^Bz_{Gy}\big) \in (-\pi, \pi] \tag{7}$$
$$\alpha = \mathrm{acos}\big({}^Bz_{Gz}\big) \in [0, \pi] \tag{8}$$
$$\theta = \mathrm{asin}\big(-{}^Bz_{Gx}\big) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \tag{9}$$
$$\phi = \mathrm{asin}\big({}^Bz_{Gy}\big) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \tag{10}$$
$$h = \mathrm{sign}\big({}^Bz_{Gz}\big) \in \{-1, 1\}. \tag{11}$$

Analysis of the geometric definition of fused yaw also reveals an alternate mathematical definition, given by

$$\psi = \begin{cases} \mathrm{wrap}\big(\mathrm{atan2}\big({}^Gz_{Bx}, -{}^Gz_{By}\big) - \gamma\big) & \text{if } \alpha \neq 0 \\ \mathrm{atan2}\big({}^Gx_{By}, {}^Gx_{Bx}\big) & \text{if } \alpha = 0 \end{cases} \tag{12}$$

where wrap is a function that wraps an angle to $(-\pi, \pi]$. It can be seen from (9–11) that ${}^B\mathbf{z}_G$ is given by a well-defined multivariate function $f_z : (\theta, \phi, h) \mapsto {}^B\mathbf{z}_G$, defined by

$${}^B\mathbf{z}_G = \Big(-\sin\theta, \sin\phi, h\sqrt{1 - \sin^2\theta - \sin^2\phi}\Big). \tag{13}$$

The domain of $f_z$, and hence the general domain of fused pitch and roll on which the fused angles representation is valid, is given by the restriction of $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \{-1, 1\}$ to $\sin^2\theta + \sin^2\phi \leq 1$. The restriction of $\hat{\mathbb{F}}$ by this *sine sum criterion* is denoted $\mathbb{F}$, the set of all valid fused angles representations. Note that the sine sum criterion is precisely equivalent to $|\theta| + |\phi| \leq \frac{\pi}{2}$. The domain of valid fused pitch and roll angles is shown graphically in Fig. 2.

The concepts of fused pitch and roll can be visualised in 3D by consideration of the level sets of $f_z$, shown in Fig. 3. The level sets are drawn as loci in 3D of ${}^B\mathbf{z}_G$ for constant fused pitch and roll. It is important to note that the plots are in the body-fixed frame $\{\mathrm{B}\}$, and not in the global fixed frame $\{\mathrm{G}\}$. Combining specifications of $\theta$ and $\phi$ into a value for ${}^B\mathbf{z}_G$ is equivalent to intersecting the corresponding cones of constant pitch and roll. Which of the two intersections is used is directly given by the $h$ parameter. Failure to intersect is equivalent to a violation of the sine sum criterion.

## IV. Conversions to Other Representations

Fused angles serve well in the analysis of body orientations, but even so, conversions to other representations are often required for mathematical computations such as rotation composition. The equations required for the conversion of a fused angles representation $F = (\psi, \theta, \phi, h) \in \mathbb{F}$ to and from tilt angles, rotation matrix and quaternion representations are presented in this section. The proofs of these equations are not difficult, but beyond the scope of this paper.

*1) Fused angles $\leftrightarrow$ Tilt angles:* The yaw parameters $\psi$ of these two representations are equal, so the conversion from fused angles to tilt angles can be completely summarised as

$$\gamma = \mathrm{atan2}(\sin\theta, \sin\phi) \tag{14}$$
$$\alpha = \mathrm{acos}\Big(h\sqrt{1 - \sin^2\theta - \sin^2\phi}\Big), \tag{15}$$

where in the latter equation one may use the trigonometric identity $1 - \sin^2\theta - \sin^2\phi = \cos(\theta + \phi)\cos(\theta - \phi)$ for numerical computation. We interestingly note from (15) that

$$\sin^2\theta + \sin^2\phi = \sin^2\alpha. \tag{16}$$

The conversion from tilt angles to fused angles is

$$\begin{aligned} \theta &= \mathrm{asin}(\sin\alpha\sin\gamma) \\ \phi &= \mathrm{asin}(\sin\alpha\cos\gamma) \end{aligned} \qquad h = \begin{cases} 1 & \text{if } \alpha \leq \frac{\pi}{2}, \\ -1 & \text{otherwise.} \end{cases} \tag{17}$$

*2) Fused angles $\leftrightarrow$ Rotation matrix:* In order to convert the fused angles vector $F$ into a rotation matrix, the equivalent tilt angles representation $(\psi, \gamma, \alpha) \in \mathbb{T}$ must first be calculated using (14–15). Based on the geometrical definition of tilt angles provided in Section III, the rotation matrix representation of $F$ can then be expressed as follows. We use the abbreviations $s_x \equiv \sin x$, $c_x \equiv \cos x$ and $\beta = \psi + \gamma$.

$$R = \begin{bmatrix} c_\gamma c_\beta + c_\alpha s_\gamma s_\beta & s_\gamma c_\beta - c_\alpha c_\gamma s_\beta & s_\alpha s_\beta \\ c_\gamma s_\beta - c_\alpha s_\gamma c_\beta & s_\gamma s_\beta + c_\alpha c_\gamma c_\beta & -s_\alpha c_\beta \\ -s_\theta & s_\phi & c_\alpha \end{bmatrix} \tag{18}$$

Note that $R_{31} = -s_\theta = -s_\alpha s_\gamma$ and $R_{32} = s_\phi = s_\alpha c_\gamma$. The inverse conversion from $R$ to $F$ follows from (7–11). If $R_m = \max\{R_{11}, R_{22}, R_{33}\}$ and $R_z = 1 - R_{11} - R_{22} + R_{33}$, then the rotation matrix to fused angles conversion can be summarised mathematically as

$$\tilde{\psi} = \begin{cases} \operatorname{atan2}(R_{21} - R_{12}, 1 + \operatorname{tr}(R)) & \text{if } \operatorname{tr}(R) \geq 0 \\ \operatorname{atan2}(R_z, R_{21} - R_{12}) & \text{if } R_m = R_{33} \\ \operatorname{atan2}(R_{32} + R_{23}, R_{13} - R_{31}) & \text{if } R_m = R_{22} \\ \operatorname{atan2}(R_{13} + R_{31}, R_{32} - R_{23}) & \text{if } R_m = R_{11} \end{cases} \tag{19}$$

$$\psi = \operatorname{wrap}(2\tilde{\psi}), \qquad \theta = \operatorname{asin}(-R_{31}), \tag{20}$$

$$\phi = \operatorname{asin}(R_{32}), \qquad h = \operatorname{sign}(R_{33}). \tag{21}$$

The corresponding tilt angles parameters can be shown to be

$$\gamma = \operatorname{atan2}(-R_{31}, R_{32}), \qquad \alpha = \operatorname{acos}(R_{33}). \tag{22}$$

*3) Fused angles $\leftrightarrow$ Quaternion:* The conversion of a tilt angles rotation $T = (\psi, \gamma, \alpha) \in \mathbb{T}$ to the corresponding quaternion representation is robustly given by

$$q = (c_{\bar{\alpha}} c_{\bar{\psi}}, s_{\bar{\alpha}} c_{\bar{\psi}+\gamma}, s_{\bar{\alpha}} s_{\bar{\psi}+\gamma}, c_{\bar{\alpha}} s_{\bar{\psi}}), \tag{23}$$

where $\bar{\alpha} = \frac{\alpha}{2}$ and $\bar{\psi} = \frac{\psi}{2}$. For fused angles, the equivalent quaternion is the normalisation of either $\tilde{q}_p$ or $\tilde{q}_n$, where

$$\tilde{q}_p = \big(c_{\bar{\psi}} C_\alpha^+, s_\phi c_{\bar{\psi}} - s_\theta s_{\bar{\psi}}, s_\phi s_{\bar{\psi}} + s_\theta c_{\bar{\psi}}, s_{\bar{\psi}} C_\alpha^+\big) \tag{24}$$

$$\tilde{q}_n = \big(s_\alpha c_{\bar{\psi}}, c_{\bar{\psi}+\gamma} C_\alpha^-, s_{\bar{\psi}+\gamma} C_\alpha^-, s_\alpha s_{\bar{\psi}}\big), \tag{25}$$

with $C_\alpha^+ = 1 + c_\alpha$ and $C_\alpha^- = 1 - c_\alpha$. It is recommended to use (24) if $h = 1$, and (25) if $h = -1$. Note that $\alpha$ never needs to be calculated, just $c_\alpha$ and $s_\alpha$. By inversion of (23), and with use of (20–21), the fused angles representation of a quaternion $q = (w, x, y, z) \in \mathbb{Q}$ can be shown to be

$$\psi = \operatorname{wrap}\big(2 \operatorname{atan2}(z, w)\big), \quad \theta = \operatorname{asin}\big(2(wy - xz)\big), \tag{26}$$

$$h = \operatorname{sign}(w^2 + z^2 - 0.5), \quad \phi = \operatorname{asin}\big(2(wx + yz)\big). \tag{27}$$

Note that this expression for $\psi$ is insensitive to the quaternion magnitude, and far more direct than an expression derived from (19–20) would be. Note also that the angle wrapping of $\psi$ is at most by a single multiple of $2\pi$. The tilt angles representation of a quaternion can be shown to be

$$\gamma = \operatorname{atan2}(wy - xz, wx + yz) \tag{28}$$

$$\alpha = \operatorname{acos}\big(2(w^2 + z^2) - 1\big). \tag{29}$$

## V. Singularity Analysis

When examining rotation representations, it is important to identify and precisely quantify any singularities. Singularities are unavoidable in any minimal parameterisation, and may occur in the form of:

- A rotation that cannot unambiguously be resolved into the required set of rotation parameters,
- A rotation for which there is no unambiguous equivalent parameterised representation,
- A rotation in the neighbourhood of which the sensitivity of the rotation to parameters map is unbounded.

The entries of a rotation matrix are a continuous function of the underlying rotation and lie in the interval $[-1, 1]$. As such, from (20–21) and the continuity of the appropriately domain-restricted arcsine function, it can be seen that the fused pitch and fused roll are continuous over the entire rotation space. Furthermore, the hemisphere parameter of the fused angles representation is uniquely and unambiguously defined over the rotation space. As a result, despite its discrete and thereby technically discontinuous nature, the hemisphere parameter is not considered to be the cause of any singularities in the fused angles representation. The fused yaw parameter, on the other hand, can be seen from (26) to have a singularity at $w = z = 0$, due to the singularity of $\operatorname{atan2}$ at $(0, 0)$. From (23), this condition can be seen to be precisely equivalent to $\alpha = \pi$, the defining equation of the set of all rotations by 180° about axes in the $\mathbf{xy}$ plane. The fused yaw singularity is a singularity of both the first and third type as per the characterisation given previously, and corresponds to an essential discontinuity of the fused yaw map. Moreover, given any fused yaw singular rotation $R$, and any neighbourhood $U$ of $R$, for every $\psi \in (-\pi, \pi]$ there exists a rotation in $U$ with a fused yaw of $\psi$. Conceptually, the fused yaw singularity can be seen as being as 'far away' from the identity rotation as possible. This is by contrast not the case for Euler angles.

The tilt angles representation trivially has the same singularity in the fused yaw as the fused angles representation. In addition to this however, from (22), the tilt axis angle $\gamma$ also has a singularity when $R_{31} = R_{32} = 0$. This corresponds to $\theta = \phi = 0$, or equivalently, $\alpha = 0$ or $\pi$—that is, either rotations of pure yaw, or rotations by 180° about axes in the $\mathbf{xy}$ plane. The tilt angle parameter $\alpha$ is continuous by (8) and the continuity of the arccosine function, and as such does not contribute any further singularities.

## VI. Results and Properties of Fused Angles

The fused angles representation possesses a remarkable number of subtle properties that turn out to be quite useful both mathematically and geometrically when working with them. One of these properties, relating to the axisymmetry of the representation, has already been stated without proof in Section II-D. Further more complex properties of fused angles, involving for example the matching of fused yaws between coordinate frames, were invoked in [1] to derive a computationally efficient algorithm to calculate instantaneous measurements of the orientation of a body from sensor data. Some of the more basic but useful properties of fused angles are presented in this section.

### A. Fundamental Properties of Fused Angles

The following fundamental properties of fused angles hold, and form a minimum set of axiomatic conditions on the fused angles parameters.

- A pure x-rotation by $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by the fused angles representation $(0, 0, \beta, 1) \in \mathbb{F}$.
- A pure y-rotation by $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by the fused angles representation $(0, \beta, 0, 1) \in \mathbb{F}$.
- A pure z-rotation by $\beta \in (-\pi, \pi]$ is given by the fused angles representation $(\beta, 0, 0, 1) \in \mathbb{F}$.
- Applying a pure z-rotation to an arbitrary fused angles rotation is purely additive in fused yaw.

Further fundamental properties of fused angles include:

- The parameter set $(\psi, \theta, \phi, h) \in \mathbb{F}$ is valid if and only if $|\theta| + |\phi| \leq \frac{\pi}{2}$, i.e. the sine sum criterion is satisfied.
- The parameter set $(\psi, \theta, \phi, h) \in \mathbb{F}$ can be put into standard form by setting $h = 1$ if $|\theta| + |\phi| = \frac{\pi}{2}$, and $\psi = 0$ if $\theta = \phi = 0$ and $h = -1$ (i.e. $\alpha = \pi$).
- Two fused angles rotations are equal if and only if their standard forms are equal.

### B. Inverse of a Fused Angles Rotation

The fused angles representation of the inverse of a rotation is intricately linked to the fused angles parameters of a rotation. This is an almost unexpected result when compared to for example Euler angles, but follows trivially from the formulas and properties presented in this paper thus far. Consider a fused angles rotation $(\psi, \theta, \phi, h)$ with an equivalent tilt angles representation $(\psi, \gamma, \alpha)$. The parameters of the inverse rotation are given by

$$\psi_{inv} = -\psi, \qquad \gamma_{inv} = \mathrm{wrap}(\psi + \gamma - \pi), \tag{30}$$

$$\alpha_{inv} = \alpha, \qquad \theta_{inv} = \mathrm{asin}\big(-\sin\alpha\sin(\psi + \gamma)\big), \tag{31}$$

$$h_{inv} = h, \qquad \phi_{inv} = \mathrm{asin}\big(-\sin\alpha\cos(\psi + \gamma)\big). \tag{32}$$

The first equation in (30) represents a remarkable property of fused yaw, one that other definitions of yaw such as ZYX Euler yaw do not satisfy. This property is referred to as negation through rotation inversion. It is worth noting that if a rotation has zero fused yaw, i.e. it is a pure tilt rotation, the inverse fused pitch and roll also satisfy the negation through rotation inversion property. That is,

$$\psi = 0 \;\Leftrightarrow\; \begin{cases} \psi_{inv} = -\psi, & \theta_{inv} = -\theta, \\ h_{inv} = h, & \phi_{inv} = -\phi. \end{cases} \tag{33}$$

### C. Characterisation of the Fused Yaw of a Quaternion

For rotations away from the singularity $\alpha = \pi$, that is, for rotations where the fused yaw is well-defined and unambiguous, inspection of (23) reveals that the z-component of a quaternion $q = (w, x, y, z) \in \mathbb{Q}$ is zero if and only if the fused yaw is zero. That is,

$$z = 0 \;\Leftrightarrow\; \psi = 0. \tag{34}$$

Furthermore, it can be seen that the quaternion corresponding to the fused yaw of the rotation can be constructed by zeroing the x and y-components of $q$ and renormalising. That is,

$$q_{yaw} = \tfrac{1}{\sqrt{w^2+z^2}}(w, 0, 0, z). \tag{35}$$

This leads to one way of removing the fused yaw component of a quaternion—something that is a surprisingly common operation—using the expression

$$q_{tilt} = q_{yaw}^{*}q = \tfrac{1}{\sqrt{w^2+z^2}}\Big(wq + z(z, y, -x, -w)\Big). \tag{36}$$

The fused yaw can also be computed using (26) and manually removed. Equations (35–36) fail only if $w = z = 0$, which is precisely equivalent to $\alpha = \pi$, the fused yaw singularity.

## VII. Conclusions

Two novel ways of parameterising a rotation were formally introduced in this paper. The main contribution of these, the fused angles representation, was developed to be able to describe a rotation in a way that yields insight on the components of the rotation in each of the three major planes of the Euclidean space. These components of rotation were termed the fused yaw, fused pitch and fused roll. The second introduced parameterisation, tilt angles, was defined solely as an intermediate representation between fused angles and other existing representations. Nevertheless, the tilt angles representation proves to be visually, conceptually and mathematically useful. Many properties of the fused angles and tilt angles representations were derived, often in highlight of their simplicity, and the relations of these two representations to other commonly used representations were explicitly given. The computational efficiency of the two representations can be seen by inspection of [6]. Due to their many special properties, fused angles fill a niche in the area of rotation parameterisation that is left vacant by alternative constructs such as Euler angles and quaternions, and are expected to yield valuable information and results, in particular in applications that involve balance.

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