

# Fused Angles: A Representation of Body Orientation for Balance

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**Abstract**—The parameterisation of rotations in three dimensional Euclidean space is an area of applied mathematics that has long been studied, dating back to the original works of Euler in the 18<sup>th</sup> century. As such, many ways of parameterising a rotation have been developed over the years. Motivated by the task of representing the orientation of a balancing body, the fused angles parameterisation is developed and introduced in this paper. This novel representation is carefully defined both mathematically and geometrically, and thoroughly investigated in terms of the properties it possesses, and how it relates to other existing representations. A second intermediate representation, tilt angles, is also introduced as a natural consequence thereof.

## I. INTRODUCTION

Numerous ways of representing a rotation in three-dimensional Euclidean space have been developed and refined over the years. Many of these representations, also referred to as parameterisations, arose naturally from classical mathematics and have found widespread use in areas such as physics, engineering and robotics. Prominent examples of such representations include rotation matrices, quaternions and Euler angles. In this paper, a new parameterisation of the manifold of all three-dimensional rotations is proposed. This parameterisation, referred to as *fused angles*, was motivated by the analysis and control of the balance of bodies in 3D, and the shortcomings of the various existing rotation representations to describe the state of balance in an intuitive and problem-relevant way. More specifically, the advent of fused angles was to address the problem of representing the orientation of a body in an environment where there is a clear notion of what is ‘up’, defined implicitly, for example, through the presence of gravity. An orientation is just a rotation relative to some global fixed frame however, so fused angles can equally be used to represent any arbitrary three-dimensional rotation, much like Euler angles, for instance, can be used for both purposes. The shortcomings of Euler angles, however, that make them unsuitable for this balance-inspired task are discussed in detail in Section II-D.

When analysing the balance state of a body, such as for example of a humanoid robot, it is very helpful to be able to work with a parameterisation of the orientation that yields information about the components of the rotation within each of the three major planes, i.e. within the  $xy$ ,  $yz$  and  $xz$  planes (see Fig. 1). These components of rotation can be conceptually thought of as a way of simultaneously quantifying the ‘amount of rotation’ about the individual axes. It is desirable for these components to each offer a useful geometric

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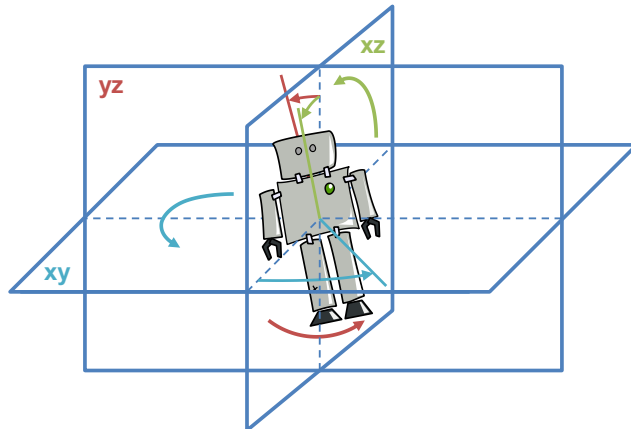


Fig. 1. Fused angles are a way of decomposing a rotation into three concurrently acting components, in such a way that it gives insight into how rotated a body is in each of the three major planes.

interpretation, and behave intuitively throughout the rotation space, most critically not sacrificing axisymmetry within the horizontal  $xy$  plane by the introduction of a clear sequential order of rotations. The notion of fusing individual rotation components in a way that avoids such an order motivated the term ‘fused angles’. Quaternions, a common choice of parameterisation in computational environments, clearly do not address these requirements, as elucidated in Section II-C.

The fused angles rotation representation has to date found a number of uses. Most recently in work published by the same authors, an attitude estimator was formulated that internally relied on the concept of fused angles [1]. The open source ROS software for the NimRo-OP humanoid robot [2], developed by the University of Bonn, also relies on the use of fused angles, most notably in the areas of state estimation and walking. Furthermore, a Matlab and Octave library [3] targeted at the numerical and computational handling of all manners of three-dimensional rotation representations, including fused angles, has been released.<sup>1</sup> This library is intended to serve as a common reference for the implementation in other programming languages of a wide range of conversion and computation functions. It is seen by the authors as a test bed to support the development of new rotation-related algorithms.

The convention that the global  $z$ -axis points in the ‘up’ direction relative to the environment is used in this paper. As mentioned previously, this accepted ‘up’ direction will almost always be defined as the antipodal direction of gravity. This ensures that definitions such as that of *fused yaw* make

<sup>1</sup> [https://github.com/AIS-Bonn/matlab\\_octave\\_rotations\\_lib](https://github.com/AIS-Bonn/matlab_octave_rotations_lib)  
Also C++ Library: [https://github.com/AIS-Bonn/rot\\_conv\\_lib](https://github.com/AIS-Bonn/rot_conv_lib)

terminological sense in consideration of the true rotation of a body relative to its environment. All derived formulas and results could easily be rewritten using an alternative convention if this were to be desired.

The contribution of this paper lies in the introduction of the novel concept of fused angles for the representation of rotations. A further contribution is the concept of tilt angles (see Section III-A), an intermediary representation that emerges naturally from the derivation of the former.

## II. REVIEW OF EXISTING ROTATION REPRESENTATIONS

Many ways of representing 3D rotations in terms of a finite set of parameters exist. Different representations have different advantages and disadvantages, and which representation is suitable for a particular application depends on a wide range of considerations. Such considerations include:

- Ease of geometric interpretation, in particular in a form that is relevant to the particular problem,
- The range of singularity-free behaviour,
- Computational efficiency in terms of common operations such as rotation composition and vector rotation,
- Mathematical convenience, in terms of numeric and algebraic complexity and manipulability, and
- Algorithmic convenience, in the sense of a representation potentially possessing properties that can conveniently be exploited for a particular algorithm.

A wide range of existing rotation representations are reviewed in this section as a basis for comparison. Due to the dimensionality of the space of 3D rotations, a minimum of three parameters is required for any such representation. A representation with exactly three parameters is referred to as *minimal*, while other representations with a greater number of parameters are referred to as *redundant*.

### A. Rotation Matrices

A rotation can be represented as a linear transformation of coordinate frame basis vectors, expressed in the form of an orthogonal matrix of unit determinant. Due to the strong link between such transformation matrices and the theory of direction cosines, the name Direction Cosine Matrix is also sometimes used. The space of all rotation matrices is called the special orthogonal group  $SO(3)$ , and is defined as

$$SO(3) = \{R \in \mathbb{R}^{3 \times 3} : R^T R = \mathbb{I}, \det(R) = 1\}. \quad (1)$$

Rotation of a vector  $\mathbf{v} \in \mathbb{R}^3$  by a rotation matrix is given by matrix multiplication. For a rotation from coordinate frame  $\{G\}$  to  $\{B\}$ , we have that

$${}^G R = [{}^G \mathbf{x}_B \quad {}^G \mathbf{y}_B \quad {}^G \mathbf{z}_B] = [{}^B \mathbf{x}_G \quad {}^B \mathbf{y}_G \quad {}^B \mathbf{z}_G]^T, \quad (2)$$

where  ${}^G \mathbf{y}_B$ , for example, is the column vector corresponding to the y-axis of frame  $\{B\}$ , expressed in the coordinates of frame  $\{G\}$ . The notation  ${}^G R$  refers to the relative rotation from  $\{G\}$  to  $\{B\}$ . With nine parameters, rotation matrices are clearly a redundant parameterisation of the rotation space. They are quite useful in that they are free of singularities and trivially expose the basis vectors of the fixed and rotated frames, but for many tasks they are not as computationally and numerically suitable as other representations.

### B. Axis-Angle and Rotation Vector Representations

By Euler's rotation theorem [4], every rotation in the three-dimensional Euclidean space  $\mathbb{R}^3$  can be expressed as a single rotation about some axis. As such, each rotation can be mapped to a pair  $(\hat{\mathbf{u}}, \theta) \in S^2 \times \mathbb{R}$ , where  $\hat{\mathbf{u}}$  is a unit vector corresponding to the axis of rotation, and  $\theta$  is the magnitude of the rotation. Note that  $S^2 = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| = 1\}$ , the 2-sphere, is the set of all unit vectors in  $\mathbb{R}^3$ . A closely related concept is that of the rotation vector, given by  $\mathbf{u} = \theta \hat{\mathbf{u}}$ , which encodes the angle of rotation as the magnitude of the vector defining the rotation axis. Both the axis-angle and rotation vector representations suffer from a general impracticality of mathematical and numerical manipulation. For example, no formula for rotation composition exists that is more direct than converting to quaternions and back. The Simultaneous Orthogonal Rotations Angle (SORA) vector, a slight reformulation of the rotation vector concept in terms of virtual angular velocities and virtual time, was presented by Tomažič and Stančin in [5]. This formulation suffers from drawbacks similar to those of the rotation vector representation, which includes a discontinuity at rotations of  $180^\circ$ , and a general lack of geometric intuitiveness.

### C. Quaternions

The set of all quaternions  $\mathbb{H}$ , and the subset  $\mathbb{Q}$  thereof of all quaternions that represent pure rotations, are defined as

$$\begin{aligned} \mathbb{H} &= \{q = (q_0, \mathbf{q}) \equiv (w, x, y, z) \in \mathbb{R}^4\}, \\ \mathbb{Q} &= \{q \in \mathbb{H} : \|q\| = 1\}. \end{aligned} \quad (3)$$

Quaternion rotations can be related to the axis-angle representation, and thereby visualised to some degree, using

$$q = (q_0, \mathbf{q}) = \left(\cos \frac{\theta}{2}, \hat{\mathbf{u}} \sin \frac{\theta}{2}\right) \in \mathbb{Q}, \quad (4)$$

where  $(\hat{\mathbf{u}}, \theta) \in S^2 \times \mathbb{R}$  is any axis-angle rotation pair, and  $q$  is the equivalent quaternion rotation. The use of quaternions to express rotations generally allows for very computationally efficient calculations, and is grounded by the well-established field of quaternion mathematics. A crucial advantage of the quaternion representation is that it is free of singularities. On the other hand however, it is not a one-to-one mapping of the special orthogonal group, as  $q$  and  $-q$  both correspond to the same rotation. The redundancy of the parameters also means that the unit magnitude constraint has to explicitly and sometimes non-trivially be enforced in numerical computations. Furthermore, no clear geometric interpretation of quaternions exists beyond the implicit relation to the axis-angle representation given in (4). For applications related to the balance of a body, where questions arise such as 'how rotated' a body is in total or within a particular major plane, the quaternion representation yields no direct insight.

### D. Euler Angles

A step in the right direction of understanding the different components of a rotation is the notion of Euler angles. In this representation, the total rotation is split into three individual elemental rotations, each about a particular coordinate frame axis. The three Euler angles  $(\alpha, \beta, \gamma)$  describing a rotation

are the successive magnitudes of these three elemental rotations. Many conventions of Euler angles exist, depending on the order in which the elemental axis rotations are chosen and whether the elemental rotations are taken to be intrinsic (about the rotating coordinate frame) or extrinsic (about the fixed coordinate frame). Extrinsic Euler angles can easily be mapped to their equivalent intrinsic Euler angles representations, and so the two types do not exhibit fundamentally different behaviour. If all three coordinate axes are used in the elemental rotations, the representation is alternatively known as Tait-Bryan angles, and the three parameters are referred to as yaw, pitch and roll, respectively. Tait-Bryan angles, although promising at first sight, do not suffice for the representation of the orientation of a body in balance-related scenarios. The main reasons for this are:

- The proximity of the gimbal lock singularity to normal working ranges, leading to unwanted artefacts due to increased local parameter sensitivity in a widened neighbourhood of the singularity,
- The fundamental requirement of an order of elemental rotations, leading to asymmetrical definitions of pitch and roll that do not correspond in behaviour, and
- The asymmetry introduced by the use of a yaw definition that depends on the projection of one of the coordinate axes onto a fixed plane, leading to unintuitive non-axisymmetric behaviour of the yaw angle.

The first listed point is a problem in real life, if for example a bipedal robot falls down, and thereby comes near the Euler angle singularity. As an example of the last of the listed points, consider the intrinsic ZYX Euler angles representation, recalling that the global z-axis points ‘upwards’ (see Section I). Consider a body in space, assumed to be in its identity orientation, and some arbitrary rotation of the body relative to its environment. It would be natural and intuitive to expect that the yaw of this relative rotation is independent of the chosen definition of the global x and y-axes. This is because the true rotation of the body is always the same, regardless of the essentially arbitrary choice of the global x and y-axes, and one would expect a well-defined yaw to be a property of the rotation, not the axis convention. This is not the case for ZYX Euler yaw however, as can be verified by counterexample with virtually any non-degenerate case. The yaw component of the fused angles representation, defined in Section III-A, can be proven to satisfy this property.

### E. Vectorial Parameterisations

Parameterisations are sometimes developed specifically to exhibit certain properties that can be exploited to increase the efficiency of an algorithm. A class of such generally more mathematical and abstract rotation representations is the family of vectorial parameterisations. Named examples of these include the Gibbs-Rodrigues parameters [6] and the Wiener-Milenković parameters [7], also known as the conformal rotation vector (CRV). Such parameterisations derive from mathematical identities such as the Euler-Rodrigues formula [8], and as such do not in general have any useful geometric interpretation, and find practical use in only very specific

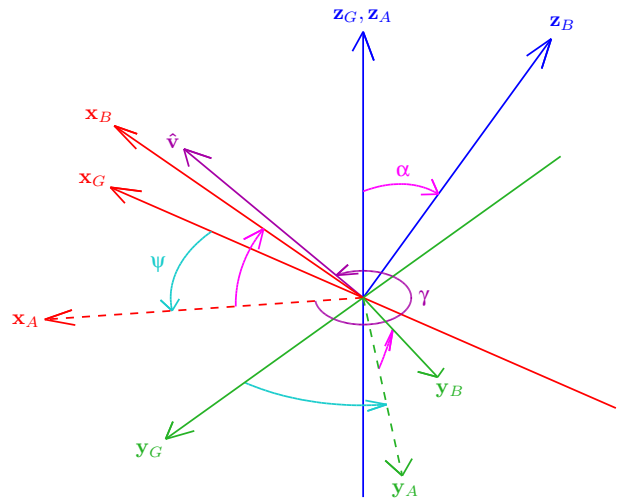


Fig. 2. Definition of the tilt rotation and tilt angles parameters of the rotation from  $\{G\}$  to  $\{B\}$ .  $\psi$  is the fused yaw,  $\gamma$  is the tilt axis angle,  $\alpha$  is the tilt angle and  $\hat{v}$  is the tilt axis. The intermediate frame  $\{A\}$  is constructed by rotating  $\{B\}$  such that  $z_B$  rotates directly onto  $z_G$ .

applications. Detailed derivations and analyses of vectorial parameterisations can be found in [6] and [7].

## III. FUSED ANGLES

Fused angles were motivated by the lack of an existing 3D rotation formalism that naturally deals with the dissolution of a complete rotation into parameters that are specifically and geometrically relevant to the balance of a body, and that does not introduce order-based asymmetry in the parameters. None of the representations discussed in Section II satisfy this property. The unwanted artefacts in the existing notions of yaw (see Section II-D) also led to the need for a more suitable, stable and axisymmetric definition of yaw.

### A. Geometric Definition of Tilt Angles

We begin by defining an intermediate rotation representation, referred to as *tilt angles*. The tilt angles parameter definitions are illustrated in Fig. 2. Note that we follow the convention that, for example,

$${}^G z_B = ({}^G z_{Bx}, {}^G z_{By}, {}^G z_{Bz}) \quad (5)$$

denotes the unit vector corresponding to the positive z-axis of a frame  $\{B\}$ , expressed in the coordinates of a frame  $\{G\}$ . The absence of a coordinate basis qualifier, such as for example in the notation ‘ $z_B$ ’, implies that a vector is by default expressed relative to the global fixed frame.

Let  $\{G\}$  denote the global fixed frame, defined with the convention that the global z-axis points upwards in the environment, as discussed in Section I. We define  $\{B\}$  to be the body-fixed coordinate frame. For an identity orientation of the body, the frames  $\{G\}$  and  $\{B\}$  should clearly coincide.

As  $z_G$  and  $z_B$  are vectors in  $\mathbb{R}^3$ , a rotation about an axis perpendicular to both vectors exists that maps  $z_G$  onto  $z_B$ . Note that this is a different condition to mapping  $\{G\}$  onto  $\{B\}$ . We choose an axis-angle representation  $(\hat{v}, \alpha)$  (see Section II-B) of this *tilt rotation* such that  $\alpha \in [0, \pi]$ . The

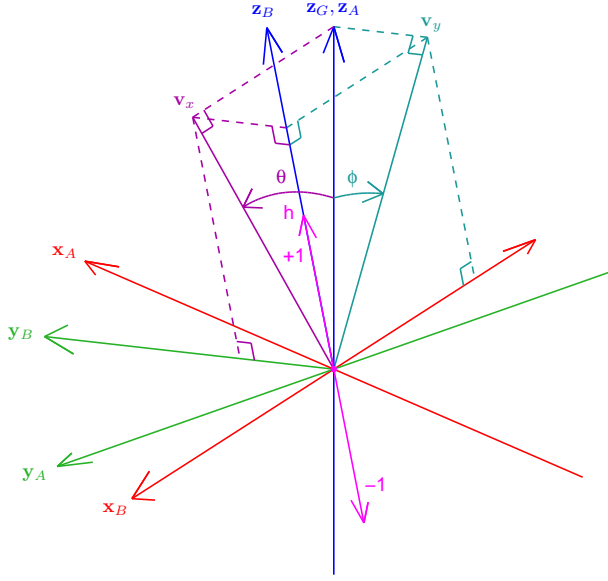


Fig. 3. Definition of the fused angles parameters  $(\theta, \phi, h)$  that describe the tilt rotation component of the rotation from  $\{G\}$  to  $\{B\}$ .  $\{A\}$  is the same intermediate frame as from the geometric definition of tilt angles (see Fig. 2). The  $\mathbf{x}_G$  and  $\mathbf{y}_G$  axes are not shown in the figure for visual simplicity.  $\theta$  is the fused pitch,  $\phi$  is the fused roll,  $h$  is the hemisphere,  $\mathbf{v}_x$  is the projection of  $\mathbf{z}_G$  onto the  $\mathbf{y}_B\mathbf{z}_B$  plane, and  $\mathbf{v}_y$  is the projection of  $\mathbf{z}_G$  onto the  $\mathbf{x}_B\mathbf{z}_B$  plane. Geometrically it can be seen that  $(\theta, \phi, h)$  depends only on the direction of  $\mathbf{z}_G$  relative to  $\{B\}$ , that is,  ${}^B\mathbf{z}_{Gz}$ .

angle  $\alpha$  is referred to as the *tilt angle* of  $\{B\}$ , and the vector  $\hat{\mathbf{v}}$  is referred to as the *tilt axis* of  $\{B\}$ . We define coordinate frame  $\{A\}$  to be the frame that results when we apply the inverse of the tilt rotation to  $\{B\}$ . By definition  $\mathbf{z}_A = \mathbf{z}_G$ , so it follows that  $\hat{\mathbf{v}}$ —and trivially also  $\mathbf{x}_A$ —must lie in the  $\mathbf{x}_A\mathbf{y}_A$  plane. The angle  $\gamma$  about  $\mathbf{z}_A$  from  $\mathbf{x}_A$  to  $\hat{\mathbf{v}}$  (see Fig. 2) is referred to as the *tilt axis angle* of  $\{B\}$ . It is easy to see that the tilt rotation from  $\{A\}$  to  $\{B\}$  is completely defined by the parameter pair  $(\gamma, \alpha)$ .

We now note that the rotation from  $\{G\}$  to  $\{A\}$  is one of pure yaw, that is, a pure  $z$ -rotation, and so define the angle  $\psi$  about  $\mathbf{z}_G$  from  $\mathbf{x}_G$  to  $\mathbf{x}_A$  (see Fig. 2) as the *fused yaw* of  $\{B\}$ . It is important to note that the choice of using the  $x$ -axes in this definition of yaw is arbitrary, and a similar definition using the  $y$ -axes would be completely equivalent. The complete tilt angles representation of the rotation from  $\{G\}$  to  $\{B\}$  is now defined as

$${}^G_B T = (\psi, \gamma, \alpha) \in (-\pi, \pi] \times (-\pi, \pi] \times [0, \pi] \equiv \mathbb{T}. \quad (6)$$

The identity tilt angles rotation is given by  $(0, 0, 0) \in \mathbb{T}$ .

It can be seen from the method of construction that all rotations possess a tilt angles representation, although it is not always necessarily unique. Most notably, when  $\alpha = 0$ , the  $\gamma$  parameter can be arbitrary with no effect.

### B. Geometric Definition of Fused Angles

To remedy the possible ambiguity in the tilt angles parameters and work towards a more robust rotation representation, we introduce the concepts of *fused pitch* and *fused roll*. For reference, the relevant fused angles parameter definitions are illustrated in Fig. 3.

Let  $\mathbf{v}_x$  and  $\mathbf{v}_y$  be the projections of the  $\mathbf{z}_G$  vector onto the body-fixed  $\mathbf{y}_B\mathbf{z}_B$  and  $\mathbf{x}_B\mathbf{z}_B$  planes respectively. We define the fused pitch of  $\{B\}$  as the angle  $\theta$  between  $\mathbf{z}_G$  and  $\mathbf{v}_x$ , of sign  $-\text{sgn}({}^B\mathbf{z}_{Gx})$ . By logical completion, the magnitude of  $\theta$  is taken to be  $\frac{\pi}{2}$  if  $\mathbf{v}_x = \mathbf{0}$ . We similarly define the fused roll of  $\{B\}$  as the angle  $\phi$  between  $\mathbf{z}_G$  and  $\mathbf{v}_y$ , of sign  $\text{sgn}({}^B\mathbf{z}_{Gy})$ . The magnitude of  $\phi$  is taken to be  $\frac{\pi}{2}$  if  $\mathbf{v}_y = \mathbf{0}$ . Conceptually, fused pitch and roll can be thought of simply as the angles between  $\mathbf{z}_G$  and the  $\mathbf{y}_B\mathbf{z}_B$  and  $\mathbf{x}_B\mathbf{z}_B$  planes respectively. Note that this definition of fused pitch and roll is invariant to the entire body-fixed frame  $\{B\}$  being yawed, as one would expect.

From inspection of the geometric definitions, it can be seen that the fused pitch and roll only uniquely specify a tilt rotation up to the  $z$ -hemisphere, that is, whether  $\mathbf{z}_B$  and  $\mathbf{z}_G$  are mutually in the same hemisphere or not. To resolve this ambiguity, the *hemisphere* of a rotation (see Fig. 3) is defined as  $\text{sign}({}^B\mathbf{z}_{Gz}) \equiv \text{sign}({}^G\mathbf{z}_{Bz})$ , where we define

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases} \quad (7)$$

Note that  $\text{sign}$  differs to the normal definition of a sign function in that  $\text{sign}(0) = 1$ , whereas  $\text{sgn}(0) = 0$ . This modified sign function is used throughout the remainder of this paper wherever clear distinction from the normal sign function is required.

Using the concept of the hemisphere of a rotation,  $(\theta, \phi, h)$  becomes a complete description of the tilt rotation component of a rotation. As such, together with the fused yaw  $\psi$ , the complete fused angles representation of the rotation from  $\{G\}$  to  $\{B\}$  can now be defined as

$${}^G_B F = (\psi, \theta, \phi, h) \in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \{-1, 1\} \equiv \hat{\mathbb{F}}. \quad (8)$$

The identity fused angles rotation is given by  $(0, 0, 0, 1) \in \hat{\mathbb{F}}$ . The  $(\theta, \phi, h)$  triplet in (8) replaces the  $(\gamma, \alpha)$  pair in (6) to define the tilt rotation component of a general rotation.

It can be observed from the geometric definitions above that the tilt rotation depends only on the direction of  $\mathbf{z}_G$  relative to frame  $\{B\}$ —that is, only on  ${}^B\mathbf{z}_{Gz}$ . This, for example, means that the bottom row of the rotation matrix  ${}^G_B R$  (representing the rotation from  $\{G\}$  to  $\{B\}$ ) can be completely identified with the tilt rotation component of that rotation. Interestingly, it can also be seen that the direction of  $\mathbf{z}_G$  relative to frame  $\{B\}$  is precisely what an accelerometer attached to the body would measure under the assumption of quasi-static conditions. In this way, accelerometer measurements can easily be mapped to measurements of  $(\theta, \phi, h)$  and/or  $(\gamma, \alpha)$ .

### C. Mathematical Definition of Fused Angles and Tilt Angles

Based on the given geometric definitions, the following expressions can be derived as an alternative mathematical definition of the tilt angles tilt rotation parameters:

$$\gamma = \text{atan2}(-{}^B\mathbf{z}_{Gx}, {}^B\mathbf{z}_{Gy}) \in (-\pi, \pi], \quad (9)$$

$$\alpha = \text{acos}({}^B\mathbf{z}_{Gz}) \in [0, \pi]. \quad (10)$$



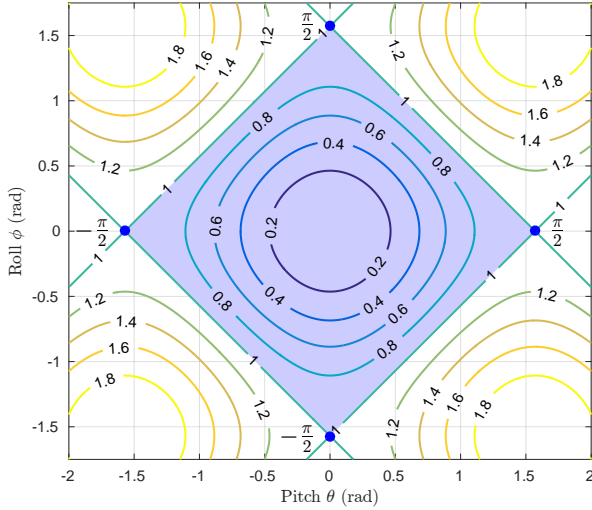


Fig. 4. Level sets of the function  $f_s(\theta, \phi) = \sin^2\theta + \sin^2\phi$ , demonstrating that the sine sum criterion  $f_s(\theta, \phi) \leq 1$  is equivalent to the simpler inequality  $|\theta| + |\phi| \leq \frac{\pi}{2}$ , indicated by the shaded region in the plot. The shaded region is the domain of  $(\theta, \phi)$  for the fused angles representation.

Similarly, alternative mathematical definitions for the fused angles tilt rotation parameters can be derived to be

$$\theta = \text{asin}(-{}^B z_{Gx}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad (11)$$

$$\phi = \text{asin}({}^B z_{Gy}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad (12)$$

$$h = \text{sign}({}^B z_{Gz}) \in \{-1, 1\}. \quad (13)$$

The analysis for the fused yaw parameter is slightly more complex, but with the use of cases one can nonetheless mathematically define it as

$$\psi = \begin{cases} \text{wrap}(\text{atan2}({}^G z_{Bx}, -{}^G z_{By}) - \gamma) & \text{if } \alpha \neq 0, \\ \text{atan2}({}^G x_{By}, {}^G x_{Bx}) & \text{if } \alpha = 0, \end{cases} \quad (14)$$

where  $\text{wrap}$  is a function that wraps an angle to  $(-\pi, \pi]$ . An alternative mathematical definition for fused yaw, namely (36), is presented later in Section IV.

It can be seen from (11–13) and the unit norm condition that  ${}^B \mathbf{z}_G$  is given by a well-defined multivariate function  $f_z : (\theta, \phi, h) \mapsto {}^B \mathbf{z}_G$ , described by

$${}^B \mathbf{z}_G = \left(-\sin\theta, \sin\phi, h\sqrt{1 - \sin^2\theta - \sin^2\phi}\right), \quad (15)$$

where for obvious reasons we must have  $\sin^2\theta + \sin^2\phi \leq 1$ . This inequality is referred to by the authors as the *sine sum criterion*, and is precisely equivalent to

$$|\theta| + |\phi| \leq \frac{\pi}{2}. \quad (16)$$

Given that by definition  $\theta, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , this equivalence can be seen by plotting the level sets of the multivariate function

$$f_s(\theta, \phi) = \sin^2\theta + \sin^2\phi, \quad (17)$$

and finding the region where  $f_s(\theta, \phi) \leq 1$ . The resulting plot is shown in Fig. 4. The domain of  $f_z$  is the restriction of  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \{-1, 1\}$  to  $|\theta| + |\phi| \leq \frac{\pi}{2}$ , and the universal set of all fused angles,  $\mathbb{F}$ , is a similar restriction of  $\hat{\mathbb{F}}$ —that is, a restriction by the sine sum criterion.

#### D. Visualisation of Fused Angles

The fused yaw parameter  $\psi$  is best visualised precisely as defined and illustrated in Fig. 2. The remaining fused angles parameters,  $(\theta, \phi, h)$ , are also well visualised based on their geometric definition shown in Fig. 3, but can alternatively be envisioned as loci of  ${}^B \mathbf{z}_G$ . Fig. 5 shows surface plots of the manifolds that are generated by independently taking the image of  $f_z(\theta, \phi, h)$  for constant fused pitch  $\theta$ , fused roll  $\phi$  and hemisphere  $h$ . The surfaces that result can be seen to be single-ended cones and hemispheres. It is important to note that the plots are in the body-fixed frame  $\{B\}$ , and not in the global fixed frame  $\{G\}$ . Fig. 5c and Fig. 5f show how combining specifications of  $\theta$ ,  $\phi$  and  $h$  acts to resolve a unique  ${}^B \mathbf{z}_G$  based on the intersection of the various hemisphere and cone loci. The failure of two cones to intersect is precisely equivalent to a violation of the sine sum criterion, and hence an invalid specification of  $\theta$  and  $\phi$ . The hemisphere parameter  $h$  essentially decides which of the two cone intersections is used for  ${}^B \mathbf{z}_G$ .

#### IV. CONVERSIONS TO OTHER REPRESENTATIONS

Fused angles serve well in the analysis of body orientations, but even so, conversions to other representations are often required for mathematical computations such as rotation composition. The equations required for the conversion of the fused angles representation  $F = (\psi, \theta, \phi, h) \in \mathbb{F}$  to and from tilt angles, rotation matrix and quaternion representations are presented in this section. Similar conversions are also provided for the tilt angles representation  $T = (\psi, \gamma, \alpha) \in \mathbb{T}$ . The proofs of the conversion equations are generally not difficult, but beyond the scope of this paper.

1) *Fused angles*  $\leftrightarrow$  *Tilt angles*: The yaw parameters  $\psi$  of these two representations are equal, so the conversion from fused angles to tilt angles is completely summarised by

$$\gamma = \text{atan2}(\sin\theta, \sin\phi), \quad (18)$$

$$\alpha = \text{acos}\left(h\sqrt{1 - \sin^2\theta - \sin^2\phi}\right), \quad (19)$$

where for numerical computation one may use the identity

$$1 - \sin^2\theta - \sin^2\phi \equiv \cos(\theta + \phi)\cos(\theta - \phi). \quad (20)$$

We interestingly note from (19) that

$$\sin^2\theta + \sin^2\phi = \sin^2\alpha. \quad (21)$$

The conversion from tilt angles to fused angles is given by

$$\begin{aligned} \theta &= \text{asin}(\sin\alpha \sin\gamma), \\ \phi &= \text{asin}(\sin\alpha \cos\gamma), \end{aligned} \quad h = \begin{cases} 1 & \text{if } \alpha \leq \frac{\pi}{2}, \\ -1 & \text{otherwise.} \end{cases} \quad (22)$$

2) *Tilt angles*  $\leftrightarrow$  *Rotation matrix*: Based on the geometric definition of tilt angles given in Section III-A, the rotation matrix equivalent to  $T = (\psi, \gamma, \alpha) \in \mathbb{T}$  can be seen to be

$$R = \begin{bmatrix} c_\gamma c_\beta + c_\alpha s_\gamma s_\beta & s_\gamma c_\beta - c_\alpha c_\gamma s_\beta & s_\alpha s_\beta \\ c_\gamma s_\beta - c_\alpha s_\gamma c_\beta & s_\gamma s_\beta + c_\alpha c_\gamma c_\beta & -s_\alpha c_\beta \\ -s_\alpha s_\gamma & s_\alpha c_\gamma & c_\alpha \end{bmatrix}, \quad (23)$$

where  $\beta = \psi + \gamma$ ,  $s_x \equiv \sin x$  and  $c_x \equiv \cos x$ .

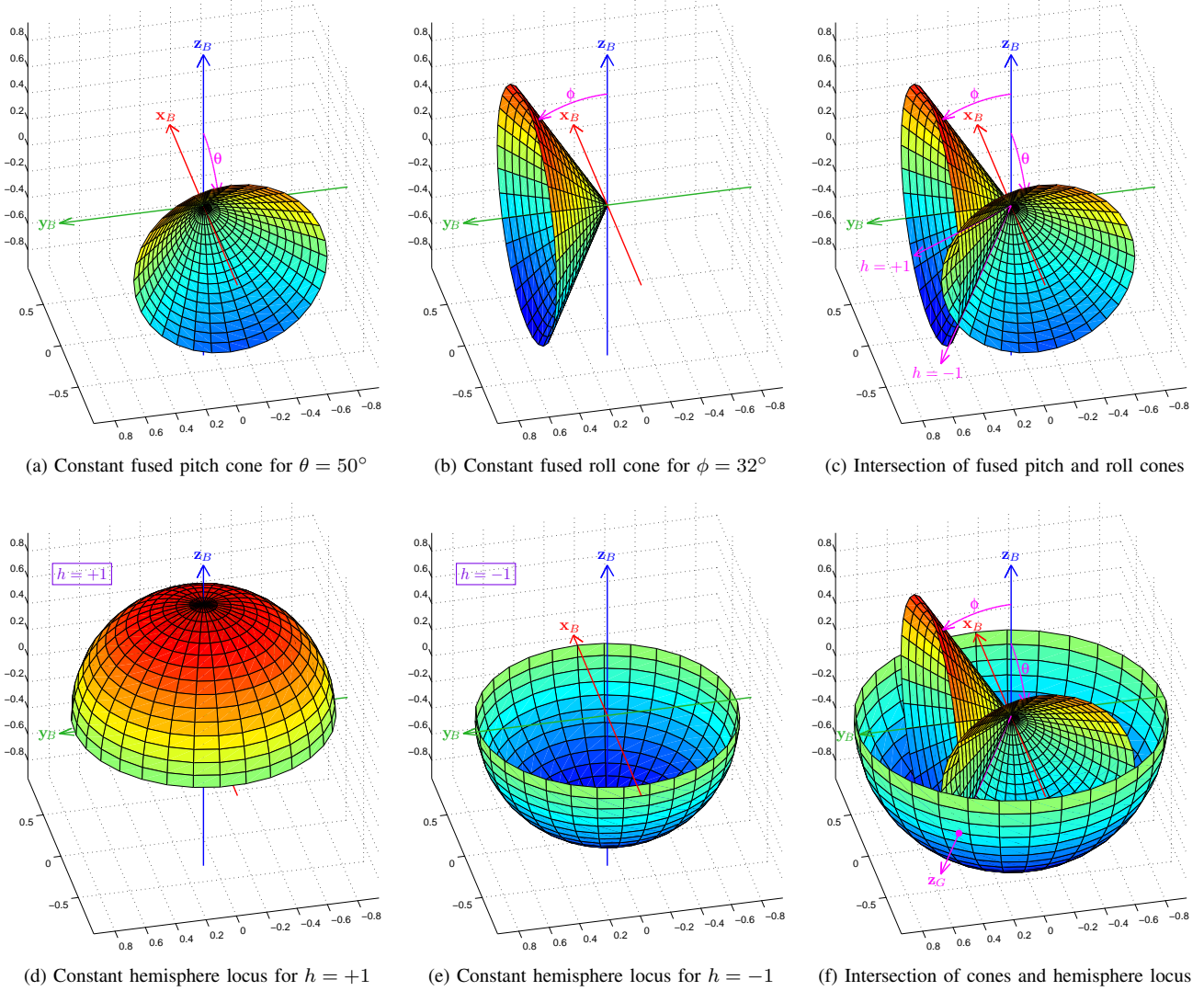


Fig. 5. Plots of the 3D locus of  ${}^B\mathbf{z}_G$  for (a) constant fused pitch  $\theta$ , (b) constant fused roll  $\phi$ , (d) constant hemisphere  $h = +1$ , and (e) constant hemisphere  $h = -1$ . (c) shows how loci of constant fused pitch and roll intersect at (at most) two points, illustrating how  $(\theta, \phi)$  specifies  ${}^B\mathbf{z}_G$  up to two possible choices. (f) shows how intersecting again with the applicable hemisphere locus uniquely resolves  ${}^B\mathbf{z}_G$ . Note that the locus of constant fused pitch in (a) is a cone with its opening facing out of the page, much like the opening of the cone of constant fused roll in (b) is facing towards the left.

The inverse conversion from a rotation matrix  $R$  to the corresponding tilt angles representation  $T$  is given by

$$\gamma = \text{atan2}(-R_{31}, R_{32}), \quad \alpha = \text{acos}(R_{33}), \quad (24)$$

where the fused yaw  $\psi$  is calculated using (26–27), as presented below for the fused angles case.

3) *Fused angles*  $\leftrightarrow$  *Rotation matrix*: The conversion of fused angles to a rotation matrix requires conversion via the tilt angles representation, using (18–19). Equation (23) can then be used with slight simplification as follows:

$$R = \begin{bmatrix} c_\gamma c_\beta + c_\alpha s_\gamma s_\beta & s_\gamma c_\beta - c_\alpha c_\gamma s_\beta & s_\alpha s_\beta \\ c_\gamma s_\beta - c_\alpha s_\gamma c_\beta & s_\gamma s_\beta + c_\alpha c_\gamma c_\beta & -s_\alpha c_\beta \\ -s_\theta & s_\phi & c_\alpha \end{bmatrix}. \quad (25)$$

The conversion from  $R$  back to  $F$  follows from (11–13). If  $R_m = \max\{R_{11}, R_{22}, R_{33}\}$  and  $R_z = 1 - R_{11} - R_{22} + R_{33}$ , then the rotation matrix to fused angles conversion is most robustly given by the following:

$$\tilde{\psi} = \begin{cases} \text{atan2}(R_{21} - R_{12}, 1 + \text{tr}(R)) & \text{if } \text{tr}(R) \geq 0 \\ \text{atan2}(R_z, R_{21} - R_{12}) & \text{if } R_m = R_{33} \\ \text{atan2}(R_{32} + R_{23}, R_{13} - R_{31}) & \text{if } R_m = R_{22} \\ \text{atan2}(R_{13} + R_{31}, R_{32} - R_{23}) & \text{if } R_m = R_{11} \end{cases} \quad (26)$$

$$\psi = \text{wrap}(2\tilde{\psi}), \quad \theta = \text{asin}(-R_{31}), \quad (27)$$

$$\phi = \text{asin}(R_{32}), \quad h = \text{sign}(R_{33}). \quad (28)$$

Although it is possible to construct a much simpler expression for  $\psi$  using (14), this is not recommended due to the resulting numerical sensitivity near  $\alpha = 0$ .

4) *Tilt angles*  $\leftrightarrow$  *Quaternion*: The conversion of a tilt angles rotation  $T = (\psi, \gamma, \alpha) \in \mathbb{T}$  to the corresponding quaternion representation is robustly given by

$$q = (c_{\bar{\alpha}} c_{\bar{\psi}}, s_{\bar{\alpha}} c_{\bar{\psi} + \gamma}, s_{\bar{\alpha}} s_{\bar{\psi} + \gamma}, c_{\bar{\alpha}} s_{\bar{\psi}}), \quad (29)$$

where  $\bar{\alpha} = \frac{1}{2}\alpha$  and  $\bar{\psi} = \frac{1}{2}\psi$ . In combination with (36) for calculating the fused yaw  $\psi$ , the inverse conversion from

quaternion  $q$  to tilt angles  $T$  is given by

$$\gamma = \text{atan2}(wy - xz, wx + yz), \quad (30)$$

$$\alpha = \text{acos}(2(w^2 + z^2) - 1). \quad (31)$$

5) *Fused angles*  $\leftrightarrow$  *Quaternion*: The conversion from fused angles to quaternions is robustly given by

$$q = \begin{cases} \frac{\tilde{q}_p}{\|\tilde{q}_p\|} & \text{if } h = 1, \\ \frac{\tilde{q}_n}{\|\tilde{q}_n\|} & \text{if } h = -1, \end{cases} \quad (32)$$

$$\tilde{q}_p = (c_{\bar{\psi}}C_{\alpha}^+, s_{\phi}c_{\bar{\psi}} - s_{\theta}s_{\bar{\psi}}, s_{\phi}s_{\bar{\psi}} + s_{\theta}c_{\bar{\psi}}, s_{\bar{\psi}}C_{\alpha}^+), \quad (33)$$

$$\tilde{q}_n = (s_{\alpha}c_{\bar{\psi}}, c_{\bar{\psi}+\gamma}C_{\alpha}^-, s_{\bar{\psi}+\gamma}C_{\alpha}^-, s_{\alpha}s_{\bar{\psi}}), \quad (34)$$

where  $C_{\alpha}^+ = 1 + c_{\alpha}$  and  $C_{\alpha}^- = 1 - c_{\alpha}$ . The respective quaternion norms are analytically given by

$$\|\tilde{q}_p\| = \sqrt{2C_{\alpha}^+} = 2c_{\bar{\alpha}}, \quad \|\tilde{q}_n\| = \sqrt{2C_{\alpha}^-} = 2s_{\bar{\alpha}}. \quad (35)$$

Note that  $\alpha$  does not need to be computed in order to evaluate (32–34), just  $c_{\alpha}$  and  $s_{\alpha}$ . These can be obtained directly from (19) and (21). Using (27–29), the fused angles representation of a quaternion  $q = (w, x, y, z) \in \mathbb{Q}$  can be shown to be

$$\psi = \text{wrap}(2 \text{atan2}(z, w)), \quad \theta = \text{asin}(2(wy - xz)), \quad (36)$$

$$h = \text{sign}(w^2 + z^2 - \frac{1}{2}), \quad \phi = \text{asin}(2(wx + yz)). \quad (37)$$

Note that this expression for  $\psi$  is insensitive to the quaternion magnitude, and far more direct than an expression derived from (26) would be. In fact, (36) can conveniently be taken as the mathematical *definition* of fused yaw. Note that the angle wrapping of  $\psi$  is at most by a single multiple of  $2\pi$ .

## V. SINGULARITY ANALYSIS

When examining rotation representations, it is important to identify and precisely quantify any singularities. Singularities are unavoidable in any minimal parameterisation, and may occur in the form of:

- (i) A rotation that cannot unambiguously be resolved into the required set of rotation parameters,
- (ii) A rotation for which there is no equivalent parameterised representation that is unambiguous,
- (iii) A rotation in the neighbourhood of which the sensitivity of the rotation to parameters map is unbounded.

The entries of a rotation matrix are a continuous function of the underlying rotation and lie in the interval  $[-1, 1]$ . As such, from (27–28) and the continuity of the appropriately domain-restricted arcsine function, it can be seen that the fused pitch and fused roll are continuous over the entire rotation space. Furthermore, the hemisphere parameter of the fused angles representation is uniquely and unambiguously defined over the rotation space. As a result, despite its discrete and thereby technically discontinuous nature, the hemisphere parameter is not considered to be the cause of any singularities in the fused angles representation.

The fused yaw parameter, on the other hand, can be seen from (36) to have a singularity at  $w = z = 0$ , due to the singularity of  $\text{atan2}$  at  $(0, 0)$ . From (29), this condition can

be seen to be precisely equivalent to  $\alpha = \pi$ , the defining equation of the set of all rotations by  $180^\circ$  about axes in the  $xy$  plane. The fused yaw singularity is a singularity of type (ii) and (iii) as per the characterisation given above, and corresponds to an essential discontinuity of the fused yaw map. Moreover, given any fused yaw singular rotation  $R$ , and any neighbourhood  $U$  of  $R$  in the rotation space  $\text{SO}(3)$ , for every  $\psi \in (-\pi, \pi]$  there exists a rotation in  $U$  with a fused yaw of  $\psi$ . Conceptually, the fused yaw singularity can be seen as being ‘far away’ from the identity rotation as possible. This is by contrast not the case for Euler angles.

The tilt angles representation trivially has the same singularity in the fused yaw as the fused angles representation. In addition to this however, from (24), the tilt axis angle  $\gamma$  also has a singularity when  $R_{31} = R_{32} = 0$ . This corresponds to  $\theta = \phi = 0$ , or equivalently,  $\alpha = 0$  or  $\pi$ —that is, either rotations of pure yaw, or rotations by  $180^\circ$  about axes in the  $xy$  plane. The tilt angle parameter  $\alpha$  is continuous by (10) and the continuity of the arccosine function, and as such does not contribute any further singularities.

## VI. RESULTS AND PROPERTIES OF FUSED ANGLES

The fused angles representation possesses a remarkable number of subtle properties that turn out to be quite useful both mathematically and geometrically when working with them. One of these properties, relating to the axisymmetry of the representation, has already been stated without proof in Section II-D. Other—more complex—properties of fused angles, involving for example the matching of fused yaws between coordinate frames, were invoked in [1] to derive a computationally efficient algorithm to calculate instantaneous measurements of the orientation of a body from sensor data. Some of the more basic but useful properties of fused angles are presented in this section.

### A. Fundamental Properties of Fused Angles

The following fundamental properties of fused angles hold, and form a minimum set of axiomatic conditions on the fused angles parameters.

- A pure x-rotation by  $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  is given by the fused angles representation  $(0, 0, \beta, 1) \in \mathbb{F}$ .
- A pure y-rotation by  $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  is given by the fused angles representation  $(0, \beta, 0, 1) \in \mathbb{F}$ .
- A pure z-rotation by  $\beta \in (-\pi, \pi]$  is given by the fused angles representation  $(\beta, 0, 0, 1) \in \mathbb{F}$ .
- Applying a pure z-rotation to an arbitrary fused angles rotation is purely additive in fused yaw.

Further fundamental properties of fused angles include:

- The parameter set  $(\psi, \theta, \phi, h) \in \mathbb{F}$  is valid if and only if  $|\theta| + |\phi| \leq \frac{\pi}{2}$ , i.e. the sine sum criterion is satisfied.
- The parameter set  $(\psi, \theta, \phi, h) \in \mathbb{F}$  can be put into standard form by setting  $h = 1$  if  $|\theta| + |\phi| = \frac{\pi}{2}$ , and  $\psi = 0$  if  $\theta = \phi = 0$  and  $h = -1$  (i.e.  $\alpha = \pi$ ).
- Two fused angles rotations are equal if and only if their standard forms are equal. Note that this identifies the  $\alpha = \pi$  rotations due to the fused yaw singularity there.

## B. Inverse of a Fused Angles Rotation

The fused angles representation of the inverse of a rotation is intricately linked to the fused angles parameters of a rotation. This is an almost unexpected result when compared to, for example, Euler angles, but follows trivially from the formulas and properties presented in this paper thus far. Consider a fused angles rotation  $(\psi, \theta, \phi, h)$  with an equivalent tilt angles representation  $(\psi, \gamma, \alpha)$ . The parameters of the inverse rotation are given by

$$\psi_{inv} = -\psi, \quad \gamma_{inv} = \text{wrap}(\psi + \gamma - \pi), \quad (38)$$

$$\alpha_{inv} = \alpha, \quad \theta_{inv} = \text{asin}(-\sin \alpha \sin(\psi + \gamma)), \quad (39)$$

$$h_{inv} = h, \quad \phi_{inv} = \text{asin}(-\sin \alpha \cos(\psi + \gamma)). \quad (40)$$

The leftmost equation in (38) represents a remarkable property of fused yaw, one that other definitions of yaw such as ZYX Euler yaw do not satisfy. This property is referred to as negation through rotation inversion. It is worth noting that if a rotation has zero fused yaw, i.e. it is a pure tilt rotation, the inverse fused pitch and roll also satisfy the negation through rotation inversion property. That is,

$$\psi = 0 \iff \begin{cases} \psi_{inv} = -\psi, & \theta_{inv} = -\theta, \\ h_{inv} = h, & \phi_{inv} = -\phi. \end{cases} \quad (41)$$

## C. Characterisation of the Fused Yaw of a Quaternion

For rotations away from the singularity  $\alpha = \pi$ , that is, for rotations where the fused yaw is well-defined and unambiguous, inspection of (29) reveals that the z-component of a quaternion  $q = (w, x, y, z) \in \mathbb{Q}$  is zero if and only if the fused yaw is zero. That is,

$$z = 0 \iff \psi = 0. \quad (42)$$

Furthermore, it can be seen that the quaternion corresponding to the fused yaw of the rotation can be constructed by zeroing the x and y-components of  $q$  and renormalising. That is,

$$q_{yaw} = \frac{1}{\sqrt{w^2+z^2}}(w, 0, 0, z). \quad (43)$$

This leads to one way of removing the fused yaw component of a quaternion—something that is a surprisingly common operation—using the expression

$$q_{tilt} = q_{yaw}^* q = \frac{1}{\sqrt{w^2+z^2}}(wq + z(z, y, -x, -w)). \quad (44)$$

The fused yaw can also be computed using (36) and manually removed. Equations (43–44) fail only if  $w = z = 0$ , which is precisely equivalent to  $\alpha = \pi$ , the fused yaw singularity.

## D. Metrics over Fused Angles

For the design of rotation space trajectories and other purposes, it is useful to be able to quantify the distance between two rotations using a metric. Assuming two fused angle rotations  $F_1$  and  $F_2$ , and their corresponding tilt angles representations  $T_1 = (\psi_1, \gamma_1, \alpha_1)$  and  $T_2 = (\psi_2, \gamma_2, \alpha_2)$ , two naturally arising metrics [9] are ( $\cdot$  is the dot product):

$$d_R(F_1, F_2) = \|\log(R_1^T R_2)\|_F = 2 \arccos(|q_1 \cdot q_2|) = \theta, \quad (45)$$

$$d_L(F_1, F_2) = 1 - \cos\left(\frac{\theta}{2}\right) = 1 - |q_1 \cdot q_2|, \quad (46)$$

where  $q_1, q_2$  are the corresponding quaternions,  $\theta$  is the angle magnitude of the relative axis-angle rotation  $(\hat{u}, \theta)$ , and

$$q_1 \cdot q_2 = c_{\bar{\alpha}_1} c_{\bar{\alpha}_2} c_{\Delta\bar{\psi}} + s_{\bar{\alpha}_1} s_{\bar{\alpha}_2} c_{\Delta\bar{\psi} + \Delta\gamma}, \quad (47)$$

where  $\Delta\bar{\psi} = \frac{1}{2}(\psi_1 - \psi_2)$ ,  $\Delta\gamma = \gamma_1 - \gamma_2$ ,  $\bar{\alpha}_1 = \frac{1}{2}\alpha_1$  and  $\bar{\alpha}_2 = \frac{1}{2}\alpha_2$ . Note that the Riemannian metric  $d_R$  relates closely to spherical linear interpolation (slerp) [10], and as such serves as a metric of first choice. Actual computation of slerp is however still most efficiently done in the quaternion space. Direct interpolation of fused angles can give unexpected results in the general case, but for two rotations in the positive hemisphere it is a viable alternative, that for many applications will produce completely satisfactory results.

## VII. CONCLUSIONS

Two novel ways of parameterising a rotation were formally introduced in this paper. The main contribution of these, the fused angles representation, was developed to be able to describe a rotation in a way that yields insight into the components of the rotation in each of the three major planes of the Euclidean space. The second parameterisation that was introduced, tilt angles, was defined as an intermediate representation between fused angles and other existing representations. Nevertheless, the tilt angles representation proves to be geometrically, conceptually and mathematically useful. Many properties of the fused angles and tilt angles representations were derived, often in highlight of their simplicity, and the relations of these two representations to other commonly used representations were explicitly given. The computational efficiency of the two representations can be seen by inspection of our open-source implementation [3]. Due to their many special properties, fused angles fill a niche in the area of rotation parameterisation that is left vacant by alternative constructs such as Euler angles and quaternions, and are expected to yield valuable information and results, in particular in applications that involve balance.

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## VIII. ADDENDUM

Some additional properties and results of rotations have been added in this section, post-publication of the paper.

### A. Rotations About Local and Global Axes

Suppose we have the rotation from  $\{G\}$  to  $\{B\}$  in one of the four representations (e.g.  ${}^G_B R$  or  ${}^G_B F$ ). Then,

- Rotation of  $\{B\}$  about a *local* axis  ${}^B \mathbf{v}$  corresponds to *post-multiplication* by the axis rotation, and,
- Rotation of  $\{B\}$  about a *global* axis  ${}^G \mathbf{v}$  corresponds to *pre-multiplication* by the axis rotation.

If frame  $\{C\}$  results from the rotation in each case, this can be expressed mathematically as:

$$\text{Local: } {}^G_C R = {}^G_B R R_{B\mathbf{v}}(\theta) \quad (48)$$

$$\text{Global: } {}^G_C R = R_{G\mathbf{v}}(\theta) {}^G_B R \quad (49)$$

### B. Referenced Rotations

Given frames  $\{A\}$  and  $\{B\}$  relative to a global frame  $\{G\}$ , the global rotation that maps  $\{A\}$  onto  $\{B\}$  is given by

$${}^G_B R = {}^G_B R {}^G_A R. \quad (50)$$

This is referred to as the rotation from frame  $\{A\}$  to  $\{B\}$ , *referenced by  $\{G\}$* . Alternative mathematical formulations for *referenced rotations* are given by

$${}^G_B R = {}^G_B R {}^G_A R^T \quad (51)$$

$$= {}^G_B R {}^A_B R {}^G_B R^T \quad (52)$$

$$= {}^G_A R {}^A_B R {}^G_A R^T. \quad (53)$$

Basic identities involving referenced rotations include

$${}^G_B R {}^G_A R = {}^G_B R, \quad (54)$$

$${}^A_B R {}^G_A R = {}^A_B R, \quad (55)$$

$${}^G_A R^T {}^G_B R {}^G_A R = {}^A_B R, \quad (56)$$

$${}^G_B R^T {}^G_A R {}^G_B R = {}^A_B R. \quad (57)$$

The simplification of trivial cases is given by

$${}^G_A R = \mathbb{I}, \quad (58)$$

$${}^G_B R = {}^G_B R, \quad (59)$$

$${}^G_A R = {}^G_A R. \quad (60)$$

Composition of referenced rotations is given by

$${}^G_C R = {}^G_B R {}^G_A R. \quad (61)$$

A change of referenced frame is given by

$${}^H_B R = {}^H_G R {}^G_B R {}^H_G R^T. \quad (62)$$

All of these definitions and results clearly also have equivalent forms in the other rotation representations. While normal rotation identities can often be thought of as ‘cancelling’ frames along the forward diagonal of successive terms, the referenced rotation identities can often be thought of as ‘cancelling’ frames along the backward diagonal.

### C. Fundamentals of Rotation Matrices

The set of all rotation matrices is called the special orthogonal group  $SO(3)$ , and is defined as

$$SO(3) = \{R \in \mathbb{R}^{3 \times 3} : R^T R = \mathbb{I}, \det(R) = 1\}. \quad (63)$$

In particular, note that  $R$  is an orthogonal matrix, so the rows and columns of  $R$  have to be mutually orthogonal unit vectors (i.e. an orthonormal basis of  $\mathbb{R}^3$ ), and

$$R^T = R^{-1}. \quad (64)$$

For a rotation from coordinate frame  $\{G\}$  to frame  $\{B\}$ ,

$${}^G_B R = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ G\mathbf{x}_B & G\mathbf{y}_B & G\mathbf{z}_B \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow {}^B \mathbf{x}_G \rightarrow \\ \leftarrow {}^B \mathbf{y}_G \rightarrow \\ \leftarrow {}^B \mathbf{z}_G \rightarrow \end{bmatrix} \quad (65)$$

$$= \begin{bmatrix} \mathbf{x}_B \cdot \mathbf{x}_G & \mathbf{y}_B \cdot \mathbf{x}_G & \mathbf{z}_B \cdot \mathbf{x}_G \\ \mathbf{x}_B \cdot \mathbf{y}_G & \mathbf{y}_B \cdot \mathbf{y}_G & \mathbf{z}_B \cdot \mathbf{y}_G \\ \mathbf{x}_B \cdot \mathbf{z}_G & \mathbf{y}_B \cdot \mathbf{z}_G & \mathbf{z}_B \cdot \mathbf{z}_G \end{bmatrix} \quad (66)$$

$$= \begin{bmatrix} \cos(\theta_{xx}) & \cos(\theta_{yx}) & \cos(\theta_{zx}) \\ \cos(\theta_{xy}) & \cos(\theta_{yy}) & \cos(\theta_{zy}) \\ \cos(\theta_{xz}) & \cos(\theta_{yz}) & \cos(\theta_{zz}) \end{bmatrix}, \quad (67)$$

where for example  ${}^G \mathbf{y}_B$  is the vector corresponding to the y-axis of frame  $\{B\}$ , expressed in the coordinates of  $\{G\}$ , and  $\theta_{zx}$  is the angle between  $\mathbf{z}_B$  and  $\mathbf{x}_G$ . Given that we are using right-handed coordinate systems, if  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  are either the rows or columns of  $R$ , then

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= 0, & \mathbf{x} \times \mathbf{y} &= \mathbf{z}, \\ \mathbf{y} \cdot \mathbf{z} &= 0, & \mathbf{y} \times \mathbf{z} &= \mathbf{x}, \\ \mathbf{z} \cdot \mathbf{x} &= 0, & \mathbf{z} \times \mathbf{x} &= \mathbf{y}. \end{aligned} \quad (68)$$

The composition of rotations is given by matrix multiplication. Rotation matrix reorthogonalisation (finding the closest valid rotation matrix  $R$  to an arbitrary matrix  $\tilde{R}$ ) is given by

$$R = \tilde{R}(\tilde{R}^T \tilde{R})^{-\frac{1}{2}}. \quad (69)$$

If  $\tilde{R}$  is a rotation matrix then clearly  $\tilde{R}^T \tilde{R} = \mathbb{I}$ , so  $R = \tilde{R}$ .  $\tilde{R}^T \tilde{R}$  is a real symmetric positive semidefinite matrix, so it is diagonalisable by a real orthogonal matrix  $P$  (i.e. for which  $P^T = P^{-1}$ ) to a diagonal matrix  $D$  with non-negative entries. That is, for such matrices  $P$  and  $D$ ,

$$\tilde{R}^T \tilde{R} = P D P^{-1}. \quad (70)$$

Let  $D^{-\frac{1}{2}}$  be the diagonal matrix with the positive inverse square roots of the diagonal of  $D$  on its diagonal. Then,

$$(\tilde{R}^T \tilde{R})^{-\frac{1}{2}} = P D^{-\frac{1}{2}} P^{-1}. \quad (71)$$

Care has to be taken regarding possibly obtaining complex numbers in the result for singular  $\tilde{R}$  matrices (i.e. for machine precision non-positive entries on the diagonal of  $D$ ). The reorthogonalisation may also result in a left-handed coordinate system ( $\det(R) = -1$ ). In each case,  $R$  can for lack of a better option just be set to  $\mathbb{I}$ .

The CCW rotation about the unit vector  $\hat{\mathbf{v}} = (v_x, v_y, v_z)$  by an angle of  $\theta$  is given by the rotation matrix

$$R_{\hat{\mathbf{v}}}(\theta) = \begin{bmatrix} v_x^2 C_\theta + c_\theta & v_x v_y C_\theta - v_z s_\theta & v_x v_z C_\theta + v_y s_\theta \\ v_x v_y C_\theta + v_z s_\theta & v_y^2 C_\theta + c_\theta & v_y v_z C_\theta - v_x s_\theta \\ v_x v_z C_\theta - v_y s_\theta & v_y v_z C_\theta + v_x s_\theta & v_z^2 C_\theta + c_\theta \end{bmatrix} \quad (72)$$

where  $C_\theta = 1 - c_\theta$ . For any rotation matrix  $R \in \text{SO}(3)$ , the transpose rotation matrix  $R^T \in \text{SO}(3)$  satisfies

$$RR^T = R^T R = \mathbb{I}, \quad (73)$$

$$(RS)^T = S^T R^T. \quad (74)$$

Rotation of a column vector  $\mathbf{v} \in \mathbb{R}^3$  by a rotation matrix is given by matrix pre-multiplication, that is

$${}^G \mathbf{v}_{rotated} = {}^G_B R {}^G \mathbf{v}. \quad (75)$$

Considering (59), this is in fact a special case of the more general expression for vector rotation, namely

$${}^G \mathbf{v}_{rotated} = {}^G_B A R {}^G \mathbf{v}. \quad (76)$$

This applies the rotation that maps {A} onto {B} to the vector  ${}^G \mathbf{v}$ , producing a vector that is also relative to {G}. If we have a frame {C}, then matrix pre-multiplication rotates the entire frame. If the resulting frame is {D}, then

$${}^G_D R = {}^G_B R {}^G_C R, \quad (77)$$

or more generally for referenced rotations,

$${}^G_D R = {}^G_B A R {}^G_C R. \quad (78)$$

An alternative way of interpreting rotation matrix pre-multiplication is as a change of reference frame, given by

$${}^G \mathbf{v} = {}^G_B R {}^B \mathbf{v}, \quad (79)$$

or for a frame {C},

$${}^G_C R = {}^G_B R {}^B_C R. \quad (80)$$

Suppose  $R \equiv {}^G_B R \in \text{SO}(3)$  is a rotation matrix, and  $\Omega$  is an arbitrary 3D angular velocity ( $\text{rad s}^{-1}$ ). Then, applying the angular velocity to  $R$  gives

$$\dot{R} = \frac{dR}{dt} = [{}^G \Omega]_{\times} R = R [{}^B \Omega]_{\times}, \quad (81)$$

where  $[\cdot]_{\times}$  is the skew-symmetric matrix representation of the cross product, given for  $\mathbf{v} = (v_x, v_y, v_z)$  by

$$[\mathbf{v}]_{\times} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}. \quad (82)$$

Note that for any vector  $\mathbf{w} \in \mathbb{R}^3$ ,

$$\mathbf{v} \times \mathbf{w} = [\mathbf{v}]_{\times} \mathbf{w}. \quad (83)$$

(81) can alternatively be written as

$$\dot{R} = \begin{bmatrix} \uparrow & & \\ G\Omega \times G\mathbf{x}_B & G\Omega \times G\mathbf{y}_B & G\Omega \times G\mathbf{z}_B \\ \downarrow & & \downarrow \end{bmatrix} \quad (84)$$

$$= \begin{bmatrix} \leftarrow & -{}^B \Omega \times {}^B \mathbf{x}_G & \rightarrow \\ \leftarrow & -{}^B \Omega \times {}^B \mathbf{y}_G & \rightarrow \\ \leftarrow & -{}^B \Omega \times {}^B \mathbf{z}_G & \rightarrow \end{bmatrix}, \quad (85)$$

so a necessary condition for  $\dot{R}$  is that all rows and columns have a dot product of zero with the corresponding rows and columns in  $R$ . Given a valid  $\dot{R}$ , the corresponding angular velocity  $\Omega$  can be read from

$$[{}^G \Omega]_{\times} = \dot{R} R^T, \quad [{}^B \Omega]_{\times} = R^T \dot{R}. \quad (86)$$

For any rotation matrix  ${}^G_B R$ , the corresponding axis-angle representation is  $(\hat{\mathbf{u}}, \theta) \in S^2 \times \mathbb{R}$  for

$$\theta = \arccos\left(\frac{1}{2}(\text{tr}(R) - 1)\right), \quad (87)$$

$$\hat{\mathbf{u}} = \frac{1}{2\sin\theta}(R_{32} - R_{23}, R_{13} - R_{31}, R_{21} - R_{12}). \quad (88)$$

If  $\theta = 0$ , then one can define  $\hat{\mathbf{u}} = \mathbf{0}$ . Strictly speaking  $\mathbf{0} \notin S^2$ , but the axis  $\hat{\mathbf{u}}$  is not uniquely defined if  $\theta = 0$ , and  $\mathbf{0}$  turns out to be a convenient value for numerical implementations. Uniformly rotating by  ${}^G_B R$  over time  $\Delta t$  corresponds to a constant angular velocity of

$${}^G \Omega = \frac{\theta}{\Delta t} \hat{\mathbf{u}}. \quad (89)$$

#### D. Fundamentals of Quaternion Rotations

The set of all quaternions  $\mathbb{H}$ , and the subset  $\mathbb{Q}$  thereof, of all quaternions that represent pure rotations, are defined as

$$\begin{aligned} \mathbb{H} &= \{q = (q_0, \mathbf{q}) \equiv (w, x, y, z) \in \mathbb{R}^4\}, \\ \mathbb{Q} &= \{q \in \mathbb{H} : \|q\| = 1\}. \end{aligned} \quad (90)$$

In particular, note that we have

$$w^2 + x^2 + y^2 + z^2 = 1. \quad (91)$$

Quaternion renormalisation (finding the closest valid quaternion rotation  $q$  to an arbitrary quaternion  $\tilde{q}$ ) is given by

$$q = \frac{\tilde{q}}{\|\tilde{q}\|}. \quad (92)$$

The CCW rotation about the unit vector  $\hat{\mathbf{v}} = (v_x, v_y, v_z)$  by an angle of  $\theta$  is given by the quaternion (taking  $w \geq 0$ )

$$\begin{aligned} q_{\hat{\mathbf{v}}}(\theta) &= \left(\cos \frac{\theta}{2}, \hat{\mathbf{v}} \sin \frac{\theta}{2}\right) \\ &= \left(\cos \frac{\theta}{2}, v_x \sin \frac{\theta}{2}, v_y \sin \frac{\theta}{2}, v_z \sin \frac{\theta}{2}\right). \end{aligned} \quad (93)$$

For  $u \in \mathbb{R}$ , and  $\theta \in [-\pi, \pi]$  (choose  $(\theta, \mathbf{v})$  that satisfies this),

$$q^u = \left(\cos \frac{u\theta}{2}, \hat{\mathbf{v}} \sin \frac{u\theta}{2}\right). \quad (94)$$

In terms of the rotations that the two quaternions represent,

$$q \equiv -q. \quad (95)$$

Given the quaternions  $p = (p_0, \mathbf{p}) = (w_1, x_1, y_1, z_1)$  and  $q = (q_0, \mathbf{q}) = (w_2, x_2, y_2, z_2)$ , multiplication is defined by

$$pq = (p_0 q_0 - \mathbf{p} \cdot \mathbf{q}, p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}) \quad (96)$$

$$= (w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2, \quad (97)$$

$$w_1 x_2 + x_1 w_2 + y_1 y_2 - z_1 y_2,$$

$$w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2,$$

$$w_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 w_2).$$

For any rotation  $q \in \mathbb{Q}$ , the conjugate  $q^*$  satisfies

$$q^* = q^{-1} = (w, -x, -y, -z), \quad (98)$$

$$qq^* = q^*q = (1, 0, 0, 0), \quad (99)$$

$$(pq)^* = q^*p^*. \quad (100)$$

The rotation of a vector  $\mathbf{v} \in \mathbb{R}^3$  by the quaternion  $q = (q_0, \mathbf{q}) \in \mathbb{Q}$  is given by  $L_q(\mathbf{v}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , defined as

$$L_q(\mathbf{v}) = q\mathbf{v}q^* = q(0, \mathbf{v})q^* \quad (101)$$

$$= (q_0^2 - \|\mathbf{q}\|^2)\mathbf{v} + 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + 2q_0(\mathbf{q} \times \mathbf{v}) \quad (102)$$

$$= \mathbf{v} + q_0\mathbf{t} + \mathbf{q} \times \mathbf{t}, \quad \text{where } \mathbf{t} = 2(\mathbf{q} \times \mathbf{v}). \quad (103)$$

Note that (103) provides the computationally most efficient method for calculating  $L_q(\mathbf{v})$ . Just like (75–80), we have

$${}^G\mathbf{v}_{rotated} = L_{Bq}({}^G\mathbf{v}), \quad (104)$$

$${}^G\mathbf{v}_{rotated} = L_{G_Aq}({}^G\mathbf{v}), \quad (105)$$

$${}^Dq = {}^Bq {}^Cq, \quad (106)$$

$${}^Dq = {}^Gq {}^Aq, \quad (107)$$

$${}^G\mathbf{v} = L_{Bq}({}^B\mathbf{v}), \quad (108)$$

$${}^Cq = {}^Bq {}^Dq. \quad (109)$$

Rotation composition works as expected:

$$L_p(L_q(\mathbf{v})) = L_{pq}(\mathbf{v}). \quad (110)$$

Suppose  $q \equiv {}^Gq \in \mathbb{Q}$  is a quaternion rotation, and  $\Omega$  is an arbitrary 3D angular velocity ( $\text{rad s}^{-1}$ ). Then, applying the angular velocity to  $q$  gives

$$\dot{q} = \frac{dq}{dt} = \frac{1}{2}(0, {}^G\Omega)q = \frac{1}{2}q(0, {}^B\Omega). \quad (111)$$

Note that  $\Omega$  is being turned into a quaternion by the addition of a zero w-component, and quaternion multiplication is being used. The output quaternion derivative  $\dot{q}$  will always be perpendicular to the original quaternion  $q$  in 4D space. That is (where  $\cdot$  is the dot product),

$$\dot{q} \cdot q = 0. \quad (112)$$

Given a quaternion derivative  $\tilde{q}$ , we first convert it into a valid quaternion derivative  $\dot{q}$  that corresponds to rotating  $q \in \mathbb{Q}$  by some angular velocity, using

$$\dot{q} = \tilde{q} - (\tilde{q} \cdot q)q. \quad (113)$$

Then, the corresponding angular velocity  $\Omega$  can be read from

$$(0, {}^G\Omega) = 2\dot{q}q^*, \quad (0, {}^B\Omega) = 2q^*\dot{q}. \quad (114)$$

(113) ensures that the w-components are indeed zero.

For any quaternion  ${}^Gq = (w, x, y, z) \in \mathbb{Q}$ , the corresponding axis-angle representation is  $(\hat{\mathbf{u}}, \theta) \in S^2 \times \mathbb{R}$  for

$$n = \|(x, y, z)\|, \quad (115)$$

$$\theta = \begin{cases} 2 \operatorname{atan2}(n, w) & \text{if } w \geq 0, \\ 2 \operatorname{atan2}(n, w) - 2\pi & \text{otherwise,} \end{cases}$$

$$\hat{\mathbf{u}} = \begin{cases} \frac{1}{n}(x, y, z) & \text{if } n \neq 0, \\ \mathbf{0} & \text{otherwise.} \end{cases} \quad (116)$$

Strictly speaking  $\mathbf{0} \notin S^2$ , but the axis  $\hat{\mathbf{u}}$  is not uniquely defined if  $n = 0$ , and  $\mathbf{0}$  turns out to be a convenient value for numerical implementations. Uniformly rotating by  ${}^Gq$  over time  $\Delta t$  corresponds to a constant angular velocity of

$${}^G\Omega = \frac{\theta}{\Delta t} \hat{\mathbf{u}}. \quad (117)$$

*E. Conversion Between Quaternions and Rotation Matrices*

Given  $q = (w, x, y, z) \in \mathbb{Q}$ , the equivalent rotation matrix is given by

$$R = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix}. \quad (118)$$

Depending on how the rotation matrix  $R$  is subsequently used, it may be necessary to coerce each of the matrix entries to  $[-1, 1]$ . Although for  $\|q\| = 1$  it is impossible in a mathematical sense for one of the entries to exceed unity in magnitude, it can happen in practice due to floating point arithmetic. In such cases, subsequent calculations such as  $\alpha = \operatorname{acos}(R_{33})$  can lead to unwanted numerical problems.

The reverse conversion is more difficult and is split into four cases, where each case corresponds to one of the four quaternion parameters being taken as the base of the conversion. Given a rotation matrix  $R \in \operatorname{SO}(3)$  with matrix entries  $R_{ij}$ , if  $\operatorname{tr}(R) \geq 0$ ,

$$r = 1 + R_{11} + R_{22} + R_{33}, \quad (119)$$

$$\tilde{q} = (r, R_{32} - R_{23}, R_{13} - R_{31}, R_{21} - R_{12}),$$

else if  $R_{33} \geq R_{22}$  and  $R_{33} \geq R_{11}$ ,

$$r = 1 - R_{11} - R_{22} + R_{33}, \quad (120)$$

$$\tilde{q} = (R_{21} - R_{12}, R_{13} + R_{31}, R_{32} + R_{23}, r),$$

else if  $R_{22} \geq R_{11}$ ,

$$r = 1 - R_{11} + R_{22} - R_{33}, \quad (121)$$

$$\tilde{q} = (R_{13} - R_{31}, R_{21} + R_{12}, r, R_{32} + R_{23}),$$

and otherwise,

$$r = 1 + R_{11} - R_{22} - R_{33}, \quad (122)$$

$$\tilde{q} = (R_{32} - R_{23}, r, R_{21} + R_{12}, R_{13} + R_{31}).$$

The output quaternion is then obtained via normalisation:

$$q = \frac{\tilde{q}}{\|\tilde{q}\|}. \quad (123)$$

This implementation of the rotation matrix to quaternion conversion is extremely robust, as it always chooses the quaternion parameter that provides the most well-conditioned problem to solve as the base of the conversion.

*F. Tilt phase space*

When working with tilt rotations (as opposed to orientations) and how to combine them, tilt angles are frequently the representation of choice. Tilt angles have a few limitations in this field of application however, such as for example that tilt rotations of more than  $180^\circ$  cannot naturally be represented. The essential discontinuity of the tilt axis angle  $\gamma$  and the non-differentiability (cusp) of the tilt angle  $\alpha$  at the identity

rotation also present numerical and algorithmic difficulties. To overcome these issues, the tilt phase space can be used instead as the representation of choice. Given the tilt angles representation  $T = (\psi, \gamma, \alpha)$  of a rotation, the equivalent  $3D$  tilt phase space representation is given by

$$\begin{aligned} P &= (p_x, p_y, p_z) = (\alpha \cos \gamma, \alpha \sin \gamma, \psi) \\ &\in \mathbb{R}^2 \times (-\pi, \pi] \equiv \mathbb{P}_3. \end{aligned} \quad (124)$$

In general, when working with tilt rotations the fused yaw component is not required. In such cases, the abridged  $2D$  tilt phase space representation can be used, given by

$$P = (p_x, p_y) = (\alpha \cos \gamma, \alpha \sin \gamma) \in \mathbb{R}^2 \equiv \mathbb{P}_2. \quad (125)$$

This is the predominant formulation of the tilt phase space that is used in general. Note that in (124–125) a domain of  $\mathbb{R}$  has been specified for  $p_x$  and  $p_y$ , to naturally be able to represent tilt rotations of more than  $180^\circ$ . The conversion from tilt phase space to tilt angles is given by

$$\psi = p_z, \quad (126)$$

$$\gamma = \text{atan2}(p_y, p_x), \quad (127)$$

$$\alpha = \sqrt{p_x^2 + p_y^2}. \quad (128)$$

It should be noted that the tilt phase variables are continuous and smooth functions of the underlying rotation, at least away from the fused yaw singularity in the case of  $p_z \equiv \psi$ . In fact, for rotations with small tilt rotation components, we have  $\sin \alpha \approx \alpha$ ,  $\sin \theta \approx \theta$  and  $\sin \phi \approx \phi$ , where  $\theta$  and  $\phi$  are the fused pitch and roll respectively, so from (22),

$$\begin{aligned} p_x &= \alpha \cos \gamma \approx \phi, \\ p_y &= \alpha \sin \gamma \approx \theta. \end{aligned} \quad (129)$$

As such, for rotations with small tilt rotation components, the  $2D$  tilt phase variables mimic fused angles, whilst far from it they continue linearly to infinity with increasing magnitude of tilt rotation. This is unlike the fused pitch and roll variables, which quickly loop around to correctly represent the resulting orientation. This emphasises the advantage of the tilt phase space for working with tilt rotations.

Although the composition of tilt rotations is not commutative and does not in general produce a tilt rotation as output, the  $2D$  tilt phase space provides a way of defining a useful and meaningful addition operator for tilt rotations that is closed, commutative and associative. The sum of two tilt rotations  $P_1, P_2 \in \mathbb{P}_2$  is defined to be

$$\begin{aligned} P_1 \oplus P_2 &= (p_{x1} + p_{x2}, p_{y1} + p_{y2}) \\ &= (\alpha_1 c_{\gamma_1} + \alpha_2 c_{\gamma_2}, \alpha_1 s_{\gamma_1} + \alpha_2 s_{\gamma_2}). \end{aligned} \quad (130)$$

If the resulting tilt rotation  $P_1 \oplus P_2$  corresponds to the tilt angles  $(\gamma_s, \alpha_s)$ , then the sum can also be written as

$$(\gamma_1, \alpha_1) \oplus (\gamma_2, \alpha_2) = (\gamma_s, \alpha_s). \quad (131)$$

It is easy to see that  $(\mathbb{P}_2, \oplus)$  is an abelian group, and defining

$$\lambda P = (\lambda p_x, \lambda p_y), \quad (132)$$

for all  $\lambda \in \mathbb{R}$ , completes  $\mathbb{P}_2$  as a vector space over  $\mathbb{R}$ . This is referred to as the *vector space of tilt rotations*.

### G. Spherical Linear Interpolation (Slerp)

Spherical linear interpolation (slerp) is a way of interpolating rotations that is torque-minimal (shortest path through the space of rotations, i.e following a geodesic) and constant velocity (constant speed in following the path). Given quaternions  $q_0, q_1 \in \mathbb{Q}$  and  $u \in [0, 1]$ , slerp is given by

$$\text{slerp}(q_0, q_1, u) = (q_1 q_0^{-1})^u q_0 = q_0 (q_0^{-1} q_1)^u. \quad (133)$$

Refer to (94) for details on taking powers of quaternions.  $q_1 q_0^{-1}$  is the relative rotation from  $q_0$  to  $q_1$ , and taking it to the power of  $u$  is like linearly interpolating the angle parameter of the corresponding relative axis-angle representation. An alternative formulation of slerp is given by

$$\begin{aligned} \Omega &= \text{acos}(q_0 \cdot q_1), \\ \text{slerp}(q_0, q_1, u) &= \left[ \frac{\sin((1-u)\Omega)}{\sin \Omega} \right] q_0 + \left[ \frac{\sin u\Omega}{\sin \Omega} \right] q_1. \end{aligned} \quad (134)$$

Due to the duality  $q \equiv -q$ , this formulation is *only valid* if  $q_0$  and  $q_1$  are in the same  $4D$  hemisphere as each other. That is, the hemisphere centred at either quaternion contains the other. This is equivalent to the condition that

$$q_0 \cdot q_1 \geq 0. \quad (135)$$

For numeric computations, first compute  $d = q_0 \cdot q_1$ . If  $d < 0$  then negate both  $d$  and  $q_1$ . Then,

$$\begin{aligned} \Omega &= \text{acos}(d), \\ \tilde{q} &= \begin{cases} (1-u)q_0 + uq_1, & \text{if } d \geq 1 - \epsilon, \\ \sin((1-u)\Omega)q_0 + \sin(u\Omega)q_1, & \text{otherwise,} \end{cases} \\ \text{slerp}(q_0, q_1, u) &= \frac{\tilde{q}}{\|\tilde{q}\|}. \end{aligned} \quad (136)$$

For double precision floating point numbers,  $\epsilon = 5e-9$  is appropriate. For any quaternion  $\hat{q} \in \mathbb{Q}$ ,

$$\text{slerp}(\hat{q}q_0, \hat{q}q_1, u) = \hat{q} \text{slerp}(q_0, q_1, u). \quad (137)$$

Refer to Section IX-A for a proof.

### H. Slerp Between Rotations of Equal Fused Yaw

Performing spherical linear interpolation (slerp) between two rotations of equal fused yaw always produces an output rotation of exactly the same fused yaw. As a corollary, this means that slerp between two tilt rotations always produces a tilt rotation. Therefore, tilt rotations can be cleanly interpolated, without affecting the fused yaw. As an equation,

$$\Psi(q_0) = \Psi(q_1) = \psi \implies \Psi(\text{slerp}(q_0, q_1, u)) = \psi. \quad (138)$$

Refer to Section IX-B for a proof.

### I. Format of Pure XYZ Rotations

A rotation about the x-, y- or z-axis is referred to as a *pure axis rotation*. Pure axis rotations as tilt angles are given by:

$$\begin{aligned} T_x(\beta) &= (0, 0, \beta), \\ T_y(\beta) &= (0, \frac{\pi}{2}, \beta), \\ T_z(\beta) &= (\beta, 0, 0). \end{aligned} \quad (139)$$



Pure axis rotations as fused angles are given by:

$$\begin{aligned} F_x(\beta) &= (0, 0, \beta, 1), \\ F_y(\beta) &= (0, \beta, 0, 1), \\ F_z(\beta) &= (\beta, 0, 0, 1). \end{aligned} \quad (140)$$

Pure axis rotations as quaternions are given by:

$$\begin{aligned} q_x(\beta) &= (c_{\bar{\beta}}, s_{\bar{\beta}}, 0, 0), \\ q_y(\beta) &= (c_{\bar{\beta}}, 0, s_{\bar{\beta}}, 0), \\ q_z(\beta) &= (c_{\bar{\beta}}, 0, 0, s_{\bar{\beta}}). \end{aligned} \quad (141)$$

Pure axis rotations as rotation matrices are given by:

$$\begin{aligned} R_x(\beta) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\beta & -s_\beta \\ 0 & s_\beta & c_\beta \end{bmatrix}, \\ R_y(\beta) &= \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix}, \\ R_z(\beta) &= \begin{bmatrix} c_\beta & -s_\beta & 0 \\ s_\beta & c_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (142)$$

### J. Format of Pure Tilt Rotations

Pure tilt rotations are rotations with zero fused yaw, i.e. for which  $\psi = 0$ . This results in the following formats in each of the respective rotation representations:

$$T_t = (0, \gamma, \alpha), \quad (143)$$

$$F_t = (0, \theta, \phi, h), \quad (144)$$

$$q_t = (c_{\bar{\alpha}}, s_{\bar{\alpha}}c_\gamma, s_{\bar{\alpha}}s_\gamma, 0). \quad (145)$$

Note that for the quaternion representation  $z = 0$ , and therefore  $w^2 + x^2 + y^2 = 1$ . For rotation matrices:

$$R_t = \begin{bmatrix} R_{11} & R_{12} & -R_{31} \\ R_{12} & R_{22} & -R_{32} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \quad (146)$$

$$= \begin{bmatrix} 1 - 2y^2 & 2xy & 2wy \\ 2xy & 1 - 2x^2 & -2wx \\ -2wy & 2wx & 2w^2 - 1 \end{bmatrix} \quad (147)$$

$$= \begin{bmatrix} c_\gamma^2 + c_\alpha s_\gamma^2 & c_\gamma s_\gamma (1 - c_\alpha) & s_\alpha s_\gamma \\ c_\gamma s_\gamma (1 - c_\alpha) & s_\gamma^2 + c_\alpha c_\gamma^2 & -s_\alpha c_\gamma \\ -s_\alpha s_\gamma & s_\alpha c_\gamma & c_\alpha \end{bmatrix} \quad (148)$$

$$= \begin{bmatrix} 1 - s_\gamma^2 (1 - c_\alpha) & c_\gamma s_\gamma (1 - c_\alpha) & s_\alpha s_\gamma \\ c_\gamma s_\gamma (1 - c_\alpha) & 1 - c_\gamma^2 (1 - c_\alpha) & -s_\alpha c_\gamma \\ -s_\alpha s_\gamma & s_\alpha c_\gamma & c_\alpha \end{bmatrix} \quad (149)$$

$$= \begin{bmatrix} \cdot & \cdot & s_\theta \\ \cdot & \cdot & -s_\phi \\ -s_\theta & s_\phi & h\sqrt{1 - s_\theta^2 - s_\phi^2} \end{bmatrix}. \quad (150)$$

In axis-angle format, a pure tilt rotation is a rotation by  $\alpha$  about the axis  $(c_\gamma, s_\gamma, 0)$ .

### K. Format of Fused Yaw Singularity

The fused yaw  $\psi$  is singular for  $180^\circ$  rotations about vectors in the xy-plane, that is, for  $180^\circ$  tilt rotations. These rotations are essential discontinuities, and are by convention defined to have a fused yaw of zero. The fused yaw singularity corresponds exactly to the following conditions:

$$\begin{aligned} \text{Singular } \psi &\iff \alpha = \pi \\ &\iff \theta = \phi = 0, h = -1 \\ &\iff w = z = 0 \\ &\iff R_{33} = -1 \\ &\iff {}^G z_{Bz} \equiv {}^B z_{Gz} = -1. \end{aligned} \quad (151)$$

The format of the  $\psi$  singularity in each representation is:

$$T_{sing} = (\psi, \gamma, \pi), \quad (152)$$

$$F_{sing} = (\psi, 0, 0, -1), \quad (153)$$

$$q_{sing} = (0, x, y, 0) = (0, c_\gamma, s_\gamma, 0). \quad (154)$$

Strictly speaking, for the representations above to be valid we should have  $\psi = 0$ . The use of a generic ' $\psi$ ' is to represent that the set of singular  $\psi$  rotations is invariant under arbitrary z-rotations. For rotation matrices the  $\psi$  singularity is:

$$R_{sing} = \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{12} & -R_{11} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (155)$$

$$= \begin{bmatrix} x^2 - y^2 & 2xy & 0 \\ 2xy & y^2 - x^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (156)$$

$$= \begin{bmatrix} c_\gamma^2 - s_\gamma^2 & 2c_\gamma s_\gamma & 0 \\ 2c_\gamma s_\gamma & s_\gamma^2 - c_\gamma^2 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (157)$$

Note in (155) that  $R_{11}^2 + R_{12}^2 = 1$ , and in particular that  $x^2 + y^2 = 1$  in (156).

### L. Format of Rotation Inverses

Given a tilt angles rotation, the inverse is given by

$$T_{inv} = (-\psi, \text{wrap}(\psi + \gamma - \pi), \alpha) \quad (158)$$

$$= (-\psi, \psi + \gamma, -\alpha), \quad (159)$$

where (159) is a shortcut if the domain of the tilt angles parameters is not important. Given a fused angles rotation, the inverse is given by

$$\begin{aligned} F_{inv} &= (-\psi, \theta_{inv}, \phi_{inv}, h), \\ \theta_{inv} &= -\text{asin}(c_\psi s_\theta + s_\psi s_\phi), \\ \phi_{inv} &= \text{asin}(s_\psi s_\theta - c_\psi s_\phi). \end{aligned} \quad (160)$$

In terms of the tilt angles parameters  $(\psi, \gamma, \alpha)$ ,  $\theta_{inv}$  and  $\phi_{inv}$  are equivalently given by

$$\begin{aligned} \theta_{inv} &= -\text{asin}(s_\alpha s_{\psi+\gamma}), \\ \phi_{inv} &= -\text{asin}(s_\alpha c_{\psi+\gamma}). \end{aligned} \quad (161)$$

For the special case of a pure tilt rotation, i.e. zero fused yaw, the inverse is given by

$$\psi = 0 \iff F_{inv} = (0, -\theta, -\phi, h). \quad (162)$$

Given a quaternion or rotation matrix, the corresponding inverse rotations are given by

$$\begin{aligned} q_{inv} &= q^*, \\ R_{inv} &= R^T. \end{aligned} \quad (163)$$

### M. Rotation Decomposition into Fused Yaw and Tilt Rotation

Rotations can be broken up into their fused yaw and tilt rotation components, as follows:

$$T = T_f \circ T_t = T(\psi, 0, 0) \circ T(0, \gamma, \alpha), \quad (164)$$

$$F = F_f \circ F_t = F(\psi, 0, 0, 1) \circ F(0, \theta, \phi, h), \quad (165)$$

$$q = q_f q_t = (c_{\bar{\psi}}, 0, 0, s_{\bar{\psi}})(c_{\bar{\alpha}}, s_{\bar{\alpha}} c_{\gamma}, s_{\bar{\alpha}} s_{\gamma}, 0). \quad (166)$$

Also, for rotation matrices:

$$\begin{aligned} R &= R_f R_t, \\ &= \begin{bmatrix} c_{\psi} & -s_{\psi} & 0 \\ s_{\psi} & c_{\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\gamma}^2 + c_{\alpha} s_{\gamma}^2 & c_{\gamma} s_{\gamma} (1 - c_{\alpha}) & s_{\alpha} s_{\gamma} \\ c_{\gamma} s_{\gamma} (1 - c_{\alpha}) & s_{\gamma}^2 + c_{\alpha} c_{\gamma}^2 & -s_{\alpha} c_{\gamma} \\ -s_{\alpha} s_{\gamma} & s_{\alpha} c_{\gamma} & c_{\alpha} \end{bmatrix}. \end{aligned} \quad (167)$$

Note that  $T_f$ ,  $F_f$ ,  $q_f$  and  $R_f$  are by definition all pure z-rotations, so we have that

$$\begin{aligned} T_f &= T_z(\psi), & F_f &= F_z(\psi), \\ q_f &= q_z(\psi), & R_f &= R_z(\psi). \end{aligned} \quad (168)$$

### N. Rotation Composition from Fused Yaw and Tilt Rotation

Given two frames, {G} and {H}, in general there is a unique frame {B} that has a desired fused yaw  ${}^G\psi$  relative to {G}, and a desired tilt rotation component  ${}^Hq_t$  relative to {H}. Cases where there are multiple solutions are discussed later. Suppose we are given  ${}^G\psi$ ,  ${}^Gq = (w_G, x_G, y_G, z_G)$  and any rotation  ${}^Hq = (w_C, x_C, y_C, z_C)$  that has the required tilt rotation component  ${}^Hq_t$  relative to {H}. The fused yaw of  ${}^Hq$  relative to {H} is irrelevant, and  ${}^Hq$  can often be calculated directly from a tilt angles specification  $T(0, \gamma, \alpha)$ . We first calculate the cross terms

$$\begin{aligned} a &= x_G x_C + y_G y_C, & b &= x_G y_C - y_G x_C, \\ c &= w_G z_C + z_G w_C, & d &= w_G w_C - z_G z_C. \end{aligned} \quad (169)$$

Then, using the abbreviated notation  $c_{\bar{\psi}} = \cos(\frac{1}{2}{}^G\psi)$  and  $s_{\bar{\psi}} = \sin(\frac{1}{2}{}^G\psi)$ , we compute the following terms:

$$\begin{aligned} A &= d - a, & B &= b - c, \\ C &= b + c, & D &= d + a, \\ G &= Dc_{\bar{\psi}} - Bs_{\bar{\psi}}, & H &= As_{\bar{\psi}} - Cc_{\bar{\psi}}. \end{aligned} \quad (170)$$

The required yaw rotation relative to {H} from {C} to {B} is then

$$F = \sqrt{G^2 + H^2}, \quad (171)$$

$${}^Hq = \begin{cases} (\frac{G}{F}, 0, 0, \frac{H}{F}), & \text{if } F \neq 0, \\ (1, 0, 0, 0), & \text{otherwise.} \end{cases} \quad (172)$$

Relative to frames {H} and {G}, frame {B} is then given by either of the expressions

$$\frac{H}{B}q = \frac{H}{B}q \frac{H}{C}q, \quad (173)$$

$$\frac{G}{B}q = \frac{G}{H}q \frac{H}{B}q \frac{H}{C}q. \quad (174)$$

The derivation of the above equations comes from setting  ${}^Hq = (c_{\bar{\psi}}, 0, 0, s_{\bar{\psi}})$ , expanding (174), and solving

$$\Psi(\frac{G}{B}q) = \frac{G}{C}q. \quad (175)$$

There are multiple solutions if  $F = 0$ , which is the case when the following two conditions hold simultaneously:

$$\alpha_G + \alpha_C = \pi, \quad (176)$$

$${}^G\psi = 2 \operatorname{atan2}(C, A). \quad (177)$$

Computationally, there is also a problem with  $F = 0$  when  $\frac{G}{H}q$  and  $\frac{H}{C}q$  are, either way around,  $(1, 0, 0, 0)$  and  $(0, 0, 1, 0)$ , or  $(0, 1, 0, 0)$  and  $(0, 0, 0, 1)$ . In this case there appears to be multiple solutions for every  ${}^G\psi$ , because of a fused yaw singularity. Note however that through a better choice of  $\frac{H}{C}q$ , it is possible to avoid the singularity and nonetheless successfully calculate the desired solution.

### O. Effect of Pure Z-rotations on Fused Yaw

Composition of any rotation with a pure z-rotation is additive in terms of fused yaw, irrespective whether the z-rotation is local (post-multiplication) or global (pre-multiplication):

$$\begin{aligned} \Psi(T \circ T_z(\psi_z)) &= \Psi(T_z(\psi_z) \circ T) = \Psi(T) + \psi_z \\ \Psi(F \circ F_z(\psi_z)) &= \Psi(F_z(\psi_z) \circ F) = \Psi(F) + \psi_z \\ \Psi(qq_z(\psi_z)) &= \Psi(q_z(\psi_z)q) = \Psi(q) + \psi_z \\ \Psi(RR_z(\psi_z)) &= \Psi(R_z(\psi_z)R) = \Psi(R) + \psi_z \end{aligned} \quad (178)$$

For global z-rotations, the tilt rotation also turns out to be untouched, so for example:

$$\begin{aligned} T_z(\psi_z) \circ T(\psi, \gamma, \alpha) &= T(\psi + \psi_z, \gamma, \alpha), \\ F_z(\psi_z) \circ F(\psi, \theta, \phi, h) &= F(\psi + \psi_z, \theta, \phi, h). \end{aligned} \quad (179)$$

Refer to Section IX-C for proofs.

### P. Link Between Fused Angles and Rotation Matrices

The correct and most efficient conversions between the fused angles and rotation matrix representations are given in Section IV. There are some other links between fused angles and rotation matrices however, that may be useful in other scenarios. For instance, the rotation matrix  $R$  corresponding to the fused angles rotation  $F = (\psi, \theta, \phi, h)$  can be written

$$R = \begin{bmatrix} c_{\gamma} c_{\beta} + c_{\alpha} s_{\gamma} s_{\beta} & s_{\gamma} c_{\beta} - c_{\alpha} c_{\gamma} s_{\beta} & s_{\psi} s_{\phi} + c_{\psi} s_{\theta} \\ c_{\gamma} s_{\beta} - c_{\alpha} s_{\gamma} c_{\beta} & s_{\gamma} s_{\beta} + c_{\alpha} c_{\gamma} c_{\beta} & s_{\psi} s_{\theta} - c_{\psi} s_{\phi} \\ -s_{\theta} & s_{\phi} & c_{\alpha} \end{bmatrix}, \quad (180)$$

where  $\beta = \psi + \gamma$ , and  $T = (\psi, \gamma, \alpha)$  is the corresponding tilt angles rotation. Recall at this point that

$$c_{\alpha} = h \sqrt{1 - s_{\theta}^2 - s_{\phi}^2}. \quad (181)$$

Converting  $R$  back to  $F$  yields

$$\begin{aligned} \theta &= \operatorname{asin}(-R_{31}), & \theta &= \operatorname{asin}(c_{\psi} R_{13} + s_{\psi} R_{23}), \\ \phi &= \operatorname{asin}(R_{32}), & \phi &= \operatorname{asin}(s_{\psi} R_{13} - c_{\psi} R_{23}), \\ h &= \operatorname{sign}(R_{33}). \end{aligned} \quad (182)$$

(26) remains the only robust expression for  $\psi$  based on  $R$ .

### Q. Constructing a Rotation from $\psi$ and ${}^B\mathbf{z}_G$

Suppose we are given the desired  $\psi$  and  ${}^B\mathbf{z}_G$  of a rotation, and wish to calculate the full representation of the rotation.

1) *Tilt angles*: Using (24), the tilt angles parameters are

$$\gamma = \text{atan2}(-{}^Bz_{Gx}, {}^Bz_{Gy}), \quad \alpha = \text{acos}({}^Bz_{Gz}). \quad (183)$$

This yields the tilt angles representation  $T = (\psi, \gamma, \alpha)$ .

2) *Fused angles*: Using (27–28), the fused angles are

$$\begin{aligned} \theta &= \text{asin}(-{}^Bz_{Gx}), & h &= \text{sign}({}^Bz_{Gz}). \\ \phi &= \text{asin}({}^Bz_{Gy}), \end{aligned} \quad (184)$$

This yields the fused angles representation  $F = (\psi, \theta, \phi, h)$ .

3) *Quaternion*: The w- and z-components are given by

$$\begin{aligned} N_{wz} &= \frac{1}{2}(1 + {}^Bz_{Gz}), & w &= c_{\bar{\psi}}\sqrt{N_{wz}}, \\ & & z &= s_{\bar{\psi}}\sqrt{N_{wz}}. \end{aligned} \quad (185)$$

The x- and y-components are then given by

$$\begin{aligned} \tilde{x} &= {}^Bz_{Gx}z + {}^Bz_{Gy}w, \\ \tilde{y} &= {}^Bz_{Gy}z - {}^Bz_{Gx}w, \end{aligned} \quad (186)$$

$$A = \sqrt{\frac{1 - N_{wz}}{\tilde{x}^2 + \tilde{y}^2}}, \quad \begin{aligned} x &= A\tilde{x}, \\ y &= A\tilde{y}. \end{aligned} \quad (187)$$

Being careful of the  $\psi$  singularity, the final quaternion is then

$$q = \begin{cases} (0, 1, 0, 0) & \text{if } \tilde{x} = \tilde{y} = 0, \\ (w, x, y, z) & \text{otherwise.} \end{cases} \quad (188)$$

For the special case that  $\psi = 0$ , we have

$$\begin{aligned} w &= \sqrt{N_{wz}}, & \tilde{x} &= {}^Bz_{Gy} \\ z &= 0, & \tilde{y} &= -{}^Bz_{Gx} \end{aligned} \quad (189)$$

Equations (187–188) are then used as before.

4) *Rotation matrix*: The most numerically safe way of constructing  $R$  from  $\psi$  and  ${}^B\mathbf{z}_G$  is to go via the quaternion representation. If  $\psi = 0$  however, and we are not close to the fused yaw singularity  ${}^Bz_{Gz} = -1$ , we can use

$$\begin{aligned} R_{11} &= 1 - \frac{{}^Bz_{Gx}^2}{1 + {}^Bz_{Gz}}, \\ R_{22} &= 1 - \frac{{}^Bz_{Gy}^2}{1 + {}^Bz_{Gz}}, \\ R_{12} &= -\frac{{}^Bz_{Gx}{}^Bz_{Gy}}{1 + {}^Bz_{Gz}}. \end{aligned} \quad (190)$$

The complete rotation matrix representation is then

$$R = \begin{bmatrix} R_{11} & R_{12} & -{}^Bz_{Gx} \\ R_{12} & R_{22} & -{}^Bz_{Gy} \\ {}^Bz_{Gx} & {}^Bz_{Gy} & {}^Bz_{Gz} \end{bmatrix}. \quad (191)$$

### R. Constructing a Rotation from $\psi$ and ${}^G\mathbf{z}_B$

Suppose we are given the desired  $\psi$  and  ${}^G\mathbf{z}_B$  of a rotation, and wish to calculate the full representation of the rotation.

1) *Tilt angles*: The tilt angles parameters are

$$\gamma = \text{atan2}({}^Gz_{Bx}, -{}^Gz_{By}) - \psi, \quad \alpha = \text{acos}({}^Gz_{Bz}). \quad (192)$$

This yields the tilt angles representation  $T = (\psi, \gamma, \alpha)$ .

2) *Fused angles*: The fused angles parameters are

$$\begin{aligned} \theta &= \text{asin}({}^Gz_{Bx}c_{\psi} + {}^Gz_{By}s_{\psi}), \\ \phi &= \text{asin}({}^Gz_{Bx}s_{\psi} - {}^Gz_{By}c_{\psi}), \\ h &= \text{sign}({}^Gz_{Bz}) \end{aligned} \quad (193)$$

This yields the fused angles representation  $F = (\psi, \theta, \phi, h)$ .

3) *Quaternion*: The w- and z-components are given by

$$\begin{aligned} N_{wz} &= \frac{1}{2}(1 + {}^Gz_{Bz}), & w &= c_{\bar{\psi}}\sqrt{N_{wz}}, \\ & & z &= s_{\bar{\psi}}\sqrt{N_{wz}}. \end{aligned} \quad (194)$$

The x- and y-components are then given by

$$\begin{aligned} \tilde{x} &= {}^Gz_{Bx}z - {}^Gz_{By}w, \\ \tilde{y} &= {}^Gz_{By}z + {}^Gz_{Bx}w, \end{aligned} \quad (195)$$

$$A = \sqrt{\frac{1 - N_{wz}}{\tilde{x}^2 + \tilde{y}^2}}, \quad \begin{aligned} x &= A\tilde{x}, \\ y &= A\tilde{y}. \end{aligned} \quad (196)$$

Being careful of the  $\psi$  singularity, the final quaternion is then

$$q = \begin{cases} (0, 1, 0, 0) & \text{if } \tilde{x} = \tilde{y} = 0, \\ (w, x, y, z) & \text{otherwise.} \end{cases} \quad (197)$$

For the special case that  $\psi = 0$ , we have

$$\begin{aligned} w &= \sqrt{N_{wz}}, & \tilde{x} &= -{}^Gz_{By} \\ z &= 0, & \tilde{y} &= {}^Gz_{Bx} \end{aligned} \quad (198)$$

Equations (196–197) are then used as before.

4) *Rotation matrix*: The most numerically safe way of constructing  $R$  from  $\psi$  and  ${}^G\mathbf{z}_B$  is to go via the quaternion representation. If  $\psi = 0$  however, and we are not close to the fused yaw singularity  ${}^Gz_{Bz} = -1$ , we can use

$$\begin{aligned} R_{11} &= 1 - \frac{{}^Gz_{Bx}^2}{1 + {}^Gz_{Bz}}, \\ R_{22} &= 1 - \frac{{}^Gz_{By}^2}{1 + {}^Gz_{Bz}}, \\ R_{12} &= -\frac{{}^Gz_{Bx}{}^Gz_{By}}{1 + {}^Gz_{Bz}}. \end{aligned} \quad (199)$$

The complete rotation matrix representation is then

$$R = \begin{bmatrix} R_{11} & R_{12} & {}^Gz_{Bx} \\ R_{12} & R_{22} & {}^Gz_{By} \\ -{}^Gz_{Bx} & -{}^Gz_{By} & {}^Gz_{Bz} \end{bmatrix}. \quad (200)$$

### S. Fundamentals of Intrinsic ZYX Euler Angles

Intrinsic ZYX Euler angles is a rotation representation that models a complete rotation as the combination of three successive relative rotations—first by the ZYX yaw  $\psi_E$  about the z-axis, then by the Euler pitch  $\theta_E$  about the new y-axis, and then by the Euler roll  $\phi_E$  about the newest x-axis. The complete ZYX Euler rotation from {G} to {B} is given by

$$\begin{aligned} {}^G_E &= (\psi_E, \theta_E, \phi_E) \\ &\in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi] \equiv \mathbb{E}. \end{aligned} \quad (201)$$

The representation is unique, except at *gimbal lock*, which is when  $\theta_E = \pm \frac{\pi}{2}$ . For  $\epsilon \in \mathbb{R}$ , the following equivalences hold:

$$(\psi_E, \theta_E, \phi_E) \equiv (\pi + \psi_E, \pi - \theta_E, \pi + \phi_E), \quad (202)$$

$$(\psi_E, \frac{\pi}{2}, \phi_E) \equiv (\psi_E - \epsilon, \frac{\pi}{2}, \phi_E - \epsilon), \quad (203)$$

$$(\psi_E, -\frac{\pi}{2}, \phi_E) \equiv (\psi_E - \epsilon, -\frac{\pi}{2}, \phi_E + \epsilon). \quad (204)$$

The rotation matrix  $R$  corresponding to the Euler angles rotation  $E = (\psi_E, \theta_E, \phi_E)$  is given by (the ‘ $E$ ’ subscripts have been omitted for notation brevity)

$$R = R_z(\psi_E)R_y(\theta_E)R_x(\phi_E) \quad (205)$$

$$= \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}_E. \quad (206)$$

The conversion from  $R$  back to  $E$  is given by

$$\begin{aligned} \psi_E &= \text{atan2}(R_{21}, R_{11}), \\ \theta_E &= \text{asin}(-R_{31}), \\ \phi_E &= \text{atan2}(R_{32}, R_{33}). \end{aligned} \quad (207)$$

The conversion from  $E$  to quaternion  $q$  is given by

$$\begin{aligned} q &= (c_{\bar{\phi}_E} c_{\bar{\theta}_E} c_{\bar{\psi}_E} + s_{\bar{\phi}_E} s_{\bar{\theta}_E} s_{\bar{\psi}_E}, \\ &\quad s_{\bar{\phi}_E} c_{\bar{\theta}_E} c_{\bar{\psi}_E} - c_{\bar{\phi}_E} s_{\bar{\theta}_E} s_{\bar{\psi}_E}, \\ &\quad c_{\bar{\phi}_E} s_{\bar{\theta}_E} c_{\bar{\psi}_E} + s_{\bar{\phi}_E} c_{\bar{\theta}_E} s_{\bar{\psi}_E}, \\ &\quad c_{\bar{\phi}_E} c_{\bar{\theta}_E} s_{\bar{\psi}_E} - s_{\bar{\phi}_E} s_{\bar{\theta}_E} c_{\bar{\psi}_E}). \end{aligned} \quad (208)$$

The conversion from  $q$  back to  $E$  is given by

$$\begin{aligned} \psi_E &= \text{atan2}(xy + wz, \frac{1}{2} - y^2 - z^2), \\ \theta_E &= \text{asin}(2(wy - xz)), \\ \phi_E &= \text{atan2}(yz + wx, \frac{1}{2} - x^2 - y^2). \end{aligned} \quad (209)$$

It can be seen from (207) and (209) that ZYX Euler pitch  $\theta_E$  and fused pitch  $\theta_F$  are the same mathematical quantity, so we identify them as

$$\theta_E \equiv \theta_F \equiv \theta. \quad (210)$$

Furthermore, the following relations tie together the ZYX Euler, fused and tilt angles parameters:

$$\phi_E = \text{atan2}(s_{\phi_F}, c_\alpha), \quad \gamma = \text{atan2}(s_\theta, c_\theta s_{\phi_E}), \quad (211)$$

$$\phi_F = \text{asin}(c_\theta s_{\phi_E}), \quad \alpha = \text{acos}(c_\theta c_{\phi_E}),$$

$$h = \text{sign}(c_{\phi_E}) = \begin{cases} 1 & \text{if } |\phi_E| \leq \frac{\pi}{2}, \\ -1 & \text{otherwise.} \end{cases} \quad (212)$$

Away from the fused yaw and ZYX yaw singularities, the relationship between the two yaws is given by

$$\psi_E = \text{wrap}(\psi_F + \gamma - \text{atan2}(c_\alpha s_\gamma, c_\gamma)), \quad (213)$$

$$\begin{aligned} \psi_F &= \text{wrap}(\psi_E - \text{atan2}(s_\theta, s_{\phi_F}) + \text{atan2}(s_\theta c_{\phi_E}, s_{\phi_E})) \\ &= \text{wrap}(\psi_E - \text{atan2}(s_\theta, c_\theta s_{\phi_E}) + \text{atan2}(s_\theta c_{\phi_E}, s_{\phi_E})). \end{aligned}$$

The relationship between having zero Euler and fused roll components in a rotation is given by

$$\phi_E = 0 \implies \phi_F = 0, \quad (214)$$

$$\phi_F = 0 \implies \phi_E = 0, \pi. \quad (215)$$

The implications of having zero Euler pitch are

$$\theta = 0 \implies \begin{cases} \psi_E = \psi_F, \\ \phi_E = \begin{cases} \phi_F & \text{if } h = 1, \\ \text{wrap}(\pi - \phi_F) & \text{if } h = -1. \end{cases} \end{cases} \quad (216)$$

The implications of having zero Euler roll are

$$\phi_E = 0 \implies \psi_E = \psi_F. \quad (217)$$

### T. Axisymmetry of the Fused Angles Representation

When using the fused angles representation in balance-related applications, by design the z-axis should be chosen to point in the direction opposite to gravity. This ensures that the concepts of ‘roll’, ‘pitch’, and in particular ‘yaw’, line up with what one would intuitively expect, e.g. that yaw is a rotation about a vertical axis that changes ‘heading’ but not ‘tilt’. The choice of z-axis however still leaves one degree of rotational freedom open in the choice of the x and y-axes. More specifically, for right-handed coordinate systems, the y-axis must be a 90° clockwise rotation of the x-axis about the z-axis, but otherwise the axes can be freely chosen. The concept of *axisymmetry* refers to the property that one or more rotation parameters are either invariant to this freedom of choice in the axes, or vary in an intuitive rotational manner proportional to the choice. More carefully, axisymmetry in this context refers to the notion that the rotation parameters, for reasons of self-consistency, should be symmetrical about the well-defined z-axis.

1) *Fused Yaw*: The fused yaw is axisymmetric in the sense that it is invariant to the choice of x and y-axes. Consider a robot that is upright, and thereby considered to have an identity orientation relative to its environment. If the robot undergoes any rotation, the above statement of fused yaw axisymmetry asserts that the fused yaw of this rotation is the same no matter what choice of x and y-axes is made. This is an important and reassuring property of the fused yaw as, given that the z-axis is unambiguously defined, any concept of yaw about the z-axis should clearly be a property of the actual physical rotation, not some arbitrary choice of a virtual reference frame made solely for the purpose of mathematical analysis. It can easily be demonstrated with virtually any non-degenerate example that Euler yaw is not axisymmetric, and for different choices of axes can readily produce deviations up to 180°.

Let  $\{U\}$  be any global coordinate frame such that  $\mathbf{z}_U$  points in the direction opposite to gravity, as required, and suppose that the rotation that is undergone by the robot is given by  ${}^U_A R$ . This is a fixed physical rotation of the robot relative to its environment, so it should have a unique well-defined fused yaw according to axisymmetry. As the z-axis is fixed, every valid global coordinate system  $\{G\}$  that can be used as a reference frame to quantify  ${}^U_A R$  (including  $\{U\}$  itself) is a pure z-rotation of  $\{U\}$ . That is, for some angle  $\beta$ ,

$${}^U_G R = R_z(\beta).$$

Given any choice of  $\{G\}$ , a body-fixed frame  $\{B\}$  is attached to the robot in such a way that it coincides with  $\{G\}$  when



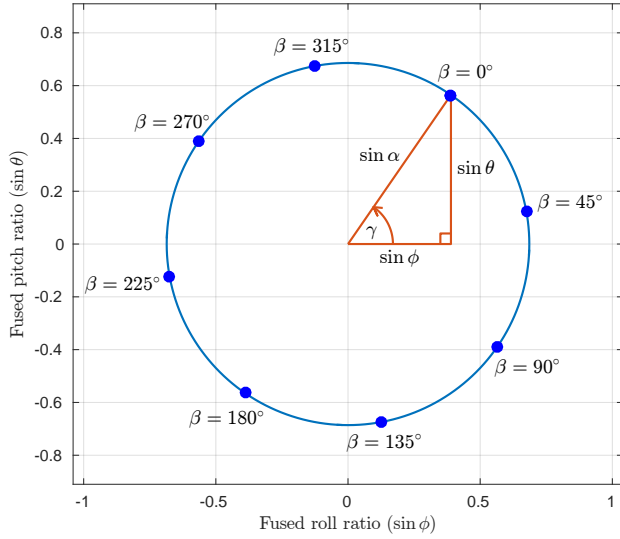


Fig. 6. Locus of the sine ratios  $(\sin \phi_\beta, \sin \theta_\beta)$  as  $\beta$  varies (i.e. for all possible choices of x and y-axes), for  $(\theta_0, \phi_0) = (0.6, 0.4)$ . The triangle demonstrates the decomposition of  $\sin \alpha$  into the quadrature sinusoid components  $\sin \theta$  and  $\sin \phi$ , and how  $\beta$  can be seen as an offset to  $\gamma$ .

the robot is initially upright, and rotates with the robot. The rotation  ${}^U_A R$  maps frame  $\{G\}$  onto frame  $\{B\}$ , so by (53)

$${}^U_A R = {}^U_G R = {}^U_G R {}^G_B R {}^U_G R^T. \quad (218)$$

Taking the fused yaw of both sides and using (178) gives

$$\begin{aligned} \Psi({}^U_A R) &= \Psi({}^U_G R {}^G_B R {}^U_G R^T) \\ &= \Psi({}^U_G R) + \Psi({}^G_B R {}^U_G R^T) \\ &= \Psi({}^U_G R) + \Psi({}^G_B R) - \Psi({}^U_G R) \\ &= \Psi({}^G_B R). \end{aligned}$$

We note that  $\Psi({}^U_A R)$  is clearly independent of the choice of  $\{G\}$ , so  $\Psi({}^G_B R)$  must also be. This demonstrates that the fused yaw of the rotation is invariant to the choice of x and y-axes (i.e.  $\{G\}$ ), as required.

To show that Euler yaw violates axisymmetry, consider

$${}^U_A R = R_x(\frac{3\pi}{4}), \quad {}^U_G R = R_z(\frac{\pi}{2}).$$

The Euler yaw of  ${}^U_A R$  is clearly 0, but from (218),

$$\begin{aligned} {}^G_B R &= {}^U_G R^T {}^U_A R {}^U_G R \\ &= E_R(\pi, -\frac{\pi}{4}, \pi), \end{aligned} \quad (219)$$

where  $E_R(\cdot)$  is notation for the rotation matrix corresponding to the given Euler angles parameters. Thus, the Euler yaw of  ${}^G_B R$  is  $\pi \neq 0$ , and as  $\{U\}$  and  $\{G\}$  are both valid choices of axes, it follows that Euler yaw cannot be axisymmetric. The asymmetry of Euler yaw is clearly illustrated for another example in the bottom plot in Fig. 9.

2) *Fused Pitch and Roll*: The fused pitch and roll are axisymmetric in the sense that their sine ratios circumscribe a uniform circle as a function of the choice of x and y-axes. That is, the locus of  $(\sin \phi, \sin \theta)$  over all possible choices of axes is a circle, and this circle is traversed uniformly as the choice varies.

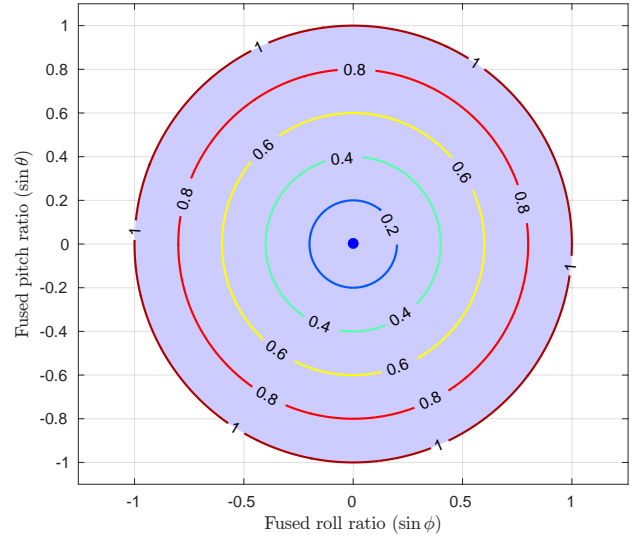


Fig. 7. Level sets of constant  $\sin \alpha$  (i.e. the sine of the magnitude of the tilt rotation component of a rotation) in the fused pitch ratio  $(\sin \theta)$  vs. fused roll ratio  $(\sin \phi)$  cartesian space. The shaded region is the valid domain of  $(\sin \phi, \sin \theta)$  for the fused angles representation. The circular nature of the plot visually illustrates the axisymmetry of fused pitch and roll.

The fused pitch  $\theta$  and fused roll  $\phi$  come together with the hemisphere  $h$  to define the tilt rotation component of a rotation. The magnitude of this tilt rotation is given by the tilt angle  $\alpha$ , and the relative direction of this tilt rotation is given by the tilt axis angle  $\gamma$ .  $\phi$  and  $\theta$  can be thought of as a way of ‘splitting up’ the action of  $\alpha$  into its orthogonal components. More precisely, the sine ratios  $\sin \phi$  and  $\sin \theta$  are in fact a decomposition of  $\sin \alpha$  into quadrature sinusoid components, as illustrated in Fig. 6 and Fig. 9, and as embodied by

$$\begin{aligned} \sin^2 \alpha &= \sin^2 \theta + \sin^2 \phi, \\ \gamma &= \text{atan2}(\sin \theta, \sin \phi). \end{aligned}$$

The axisymmetry property is equivalent to stating that the choice of x and y-axes simply results in a fixed phase shift to the quadrature components. This suggests that the nature of fused pitch and roll in expressing a tilt rotation component is more a property of the actual physical rotation than whatever arbitrary reference frame is chosen to quantify it numerically. It can easily be demonstrated that Euler pitch and roll are not axisymmetric.

Consider the same robot undergoing a rotation as in Section VIII-T.1, where we had

$${}^U_G R = R_z(\beta).$$

We now introduce the notation

$$\begin{aligned} {}^U_A R &= T_R(\psi_0, \gamma_0, \alpha_0) \\ &= F_R(\psi_0, \theta_0, \phi_0, h_0), \\ {}^G_B R &= F_R(\psi_\beta, \theta_\beta, \phi_\beta, h_\beta), \end{aligned} \quad (220)$$

where  $F_R(\cdot)$  and  $T_R(\cdot)$  are respectively the rotation matrices corresponding to the given fused and tilt angles parameters. Using this notation, the axisymmetry of fused yaw is equivalent to the statement

$$\psi_\beta = \psi_0.$$

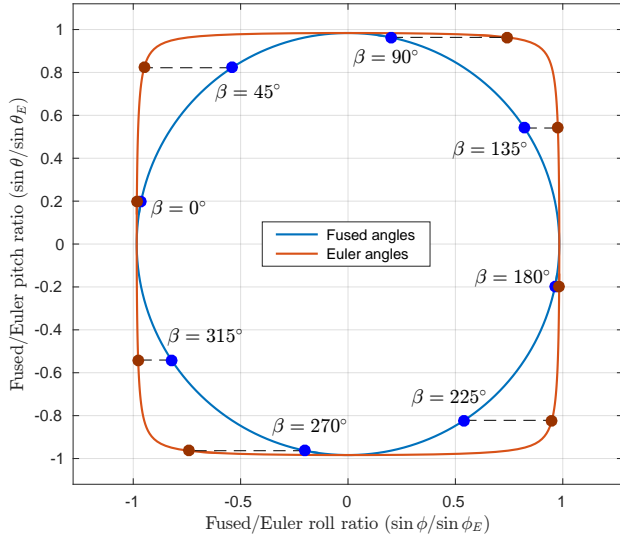


Fig. 8. Loci of the sine ratios  $(\sin \phi_\beta, \sin \theta_\beta)$  as  $\beta$  varies (i.e. for all possible choices of x and y-axes), once for fused pitch  $\theta_F$  and roll  $\phi_F$  (inner curve), and once for Euler pitch  $\theta_E$  and roll  $\phi_E$  (outer curve). The base rotation  $U_A R$  used for both loci is  $F_R(-1.2, 0.2, -1.3, -1)$ . The non-circularity of the Euler locus, as well as the non-uniformity of the associated keypoints, demonstrate the violation of axisymmetry for Euler pitch and roll.

From (219–220), (25) and (142),

$$\begin{aligned} {}^G_B R &= R_z(\beta)^T U_A R R_z(\beta) \\ &= \begin{bmatrix} \cdot & \cdot & \cdot \\ s_\beta s_{\phi_0} - c_\beta s_{\theta_0} & c_\beta s_{\phi_0} + s_\beta s_{\theta_0} & c_{\alpha_0} \end{bmatrix} \end{aligned} \quad (221)$$

where the ‘.’ entries are omitted for brevity. Using (25) to expand the left-hand side and comparing matrix entries gives

$$\begin{aligned} -s_{\theta_\beta} &= s_\beta s_{\phi_0} - c_\beta s_{\theta_0}, \\ s_{\phi_\beta} &= c_\beta s_{\phi_0} + s_\beta s_{\theta_0}. \end{aligned}$$

This can be written as the matrix equation

$$\begin{bmatrix} \sin \phi_\beta \\ \sin \theta_\beta \end{bmatrix} = \begin{bmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{bmatrix} \begin{bmatrix} \sin \phi_0 \\ \sin \theta_0 \end{bmatrix}.$$

By identifying the middle matrix as a 2D rotation matrix, this equation can be seen to be the mathematical expression of the axisymmetry of fused pitch and roll. The effect of varying  $\beta$ , and how this leads to a uniform circular locus is shown in Fig. 6. The phase shift to the quadrature components can be seen to just be  $-\beta$ .

One can observe that choosing arbitrary x and y-axes is equivalent to choosing an arbitrary  $\beta$ , which in turn is equivalent to adding an arbitrary offset to  $\gamma_0$ . In fact, it can be seen from the phase shift that

$$\gamma_\beta = \gamma_0 - \beta.$$

As such, all possible loci of sine ratios can be visualised at once by plotting the contours of constant  $\alpha$  while  $\gamma$  varies. This is equivalent to generating the level sets of constant  $\sin \alpha$  in the fused pitch ratio vs. fused roll ratio plane, the result of which is shown in Fig. 7. The axisymmetry of fused

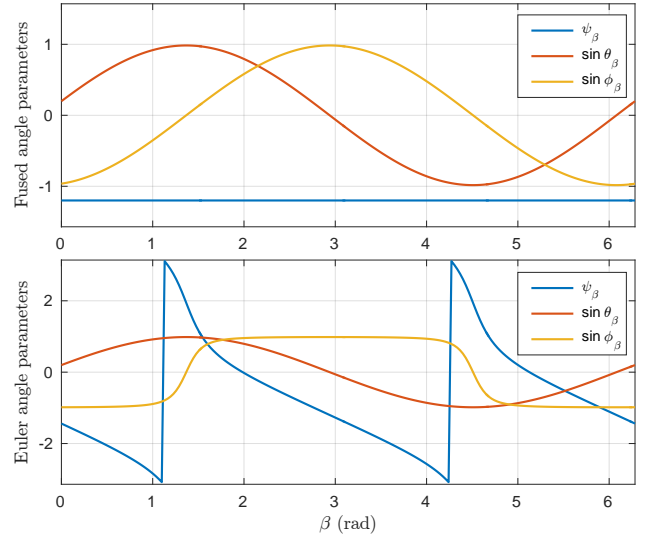


Fig. 9. Plots of pitch and roll sine ratios and yaw against  $\beta$  for both fused angles (top) and Euler angles (bottom). The base rotation  $U_A R$  used for both plots is  $F_R(-1.2, 0.2, -1.3, -1)$ , as in Fig. 8. The invariance of the fused yaw, as well as the exact quadrature nature of the fused pitch and roll can be clearly identified. The non-axisymmetry of Euler pitch and roll, and in particular the irregularity of Euler yaw, can be seen in the bottom plot.

pitch and roll can be clearly visually identified in the figure. An analogous figure for Euler angles is provided in Fig. 10.

The non-axisymmetry of all three Euler angles parameters is visualised in Fig. 8 and Fig. 9. The two figures also clearly visualise the corresponding axisymmetry of fused angles for the same base rotation  $U_A R$ . Conceptually, the problem of Euler pitch and roll is the fundamental requirement of an order of rotations. This leads to definitions of pitch and roll that do not correspond to each other in behaviour, as one then implicitly depends on the other.

3) *Hemisphere*: The hemisphere is axisymmetric in the sense that it is invariant to the choice of x and y-axes. As the (3, 3) entry of  ${}^G_B R$  is  $c_{\alpha_\beta}$ , it can be seen from (221) that

$$\alpha_\beta = \alpha_0.$$

The hemisphere in general is a function only of  $\alpha$ , as shown in (22), so it immediately follows that

$$h_\beta = h_0.$$

This demonstrates that both the tilt angle  $\alpha$  and hemisphere  $h$  are axisymmetric.

## IX. SUPPORTING PROOFS

### A. Proof of Section VIII-G

Recall from (133) that for  $q_0, q_1 \in \mathbb{Q}$  and  $u \in \mathbb{R}$ ,  $\text{slerp}$  is given by the equation

$$\text{slerp}(q_0, q_1, u) = (q_1 q_0^{-1})^u q_0 = q_0 (q_0^{-1} q_1)^u.$$

So for any  $\hat{q} \in \mathbb{Q}$ ,

$$\begin{aligned} \text{slerp}(\hat{q}q_0, \hat{q}q_1, u) &= \hat{q}q_0 ((\hat{q}q_0)^{-1} \hat{q}q_1)^u \\ &= \hat{q}q_0 (q_0^{-1} \hat{q}^{-1} \hat{q}q_1)^u \\ &= \hat{q} [q_0 (q_0^{-1} q_1)^u] \\ &= \hat{q} \text{slerp}(q_0, q_1, u). \end{aligned}$$

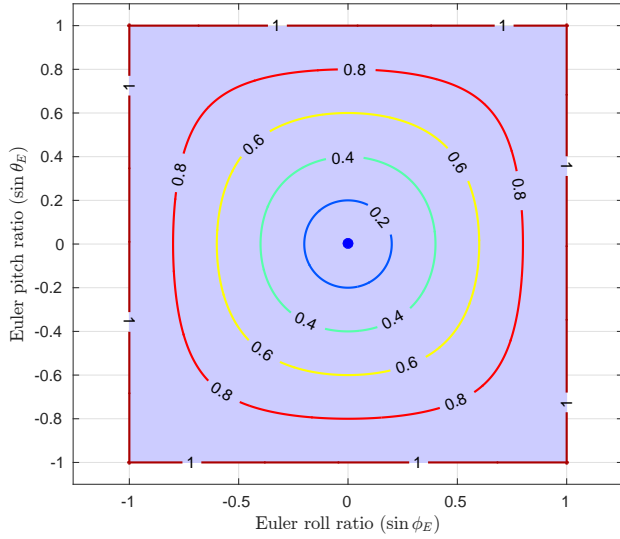


Fig. 10. Level sets of constant  $\sin \alpha$  in the Euler pitch ratio ( $\sin \theta_E$ ) vs. Euler roll ratio ( $\sin \phi_E$ ) cartesian space.

### B. Proof of Section VIII-H

If  $\Psi(q_0) = \Psi(q_1) = \psi$ , we know from Section VIII-M that

$$\left. \begin{aligned} q_0 &= q_{f0}q_{t0} \\ q_1 &= q_{f1}q_{t1} \end{aligned} \right\} \text{for } q_{f0} = q_{f1} = q_z(\psi),$$

where  $q_{t0}, q_{t1}$  are tilt rotations. Thus from (137),

$$\text{slerp}(q_0, q_1, u) = q_z(\psi) \text{slerp}(q_{t0}, q_{t1}, u).$$

If  $\text{slerp}(q_{t0}, q_{t1}, u)$  is a tilt rotation (the assertion of the result being proven), then

$$\Psi(\text{slerp}(q_0, q_1, u)) = \psi.$$

Therefore, the result holds for all  $\psi$  if it holds for tilt rotations (i.e.  $\psi = 0$ ). Suppose  $\Psi(q_0) = \Psi(q_1) = 0$ . Then, from (42) the z-components of  $q_0$  and  $q_1$  are zero. Recall from (134) that an equivalent definition of  $\text{slerp}$  is given by

$$\text{slerp}(q_0, q_1, u) = \left[ \frac{\sin((1-u)\Omega)}{\sin \Omega} \right] q_0 + \left[ \frac{\sin(u\Omega)}{\sin \Omega} \right] q_1.$$

As the output quaternion is just a linear combination of  $q_0$  and  $q_1$ , it is clear that the output quaternion must also have a zero z-component. Therefore, from (42),

$$\Psi(\text{slerp}(q_0, q_1, u)) = 0,$$

and the proof is complete.

### C. Proof of Section VIII-O

To see why (178–179) are true, consider

$$\begin{aligned} R_z(\psi_z)R &= R_z(\psi_z)R_f R_t \\ &= R_z(\psi_z)R_z(\psi)R_t \\ &= R_z(\psi + \psi_z)R_t \\ \implies \Psi(R_z(\psi_z)R) &= \psi + \psi_z. \end{aligned}$$

Note that the resulting tilt rotation component of  $R_z(\psi_z)R$  is unchanged from  $R$ , demonstrating (179). Now recall that  $\Psi(R^{-1}) = -\Psi(R)$ , define  $\tilde{R} \equiv R^{-1}$ , and consider

$$\begin{aligned} (RR_z(\psi_z))^{-1} &= R_z(\psi_z)^{-1}\tilde{R} \\ &= R_z(-\psi_z)\tilde{R}_f\tilde{R}_t \\ &= R_z(-\psi_z)R_z(-\psi)\tilde{R}_t \\ &= R_z(-\psi - \psi_z)\tilde{R}_t \\ \implies \Psi(RR_z(\psi_z)) &= -\Psi((RR_z(\psi_z))^{-1}) \\ &= -\Psi(R_z(-\psi - \psi_z)\tilde{R}_t) \\ &= \psi + \psi_z. \end{aligned}$$

The remaining results follow trivially from rotation representation conversions.