1 Einleitung

Wir beschäftigen uns mit der affinen Grassmannschen $\mathscr{G}_G$ einer reduktiven abgeschlossenen Untergruppe G der SL_n, definiert über einem algebraisch abgeschlossenen Körper k beliebiger Charakteristik. Dabei ist $\mathscr{G}_G$ ein ind-darstellbarer k-Raum, dessen k-wertige Punkte durch $G(k(\!(z)\!))/G(k[\![z]\!])$ gegeben sind. Die Konstruktion von $\mathscr{G}_G$ wird in den Abschnitten 3 und 4 detailliert ausgeführt.
Auf $\mathscr{G}_G$ operiert $G(k[\![z]\!])$ in natürlicher Weise. Die Bahnen dieser Operation werden durch die dominanten Kogewichte $\lambda \in \Lambda_G^+$ von G parametrisiert. Wir schreiben $\mathscr{G}_\lambda$ für die zu λ assoziierte Bahn.
Eine *(affine) Schubertvarietät* in $\mathscr{G}_G$ ist der Abschluss $\overline{\mathscr{G}_\lambda} \subset \mathscr{G}_G$ eines $G(k[\![z]\!])$-Orbits $\mathscr{G}_\lambda$.
Es ist bekannt, dass

$$\overline{\mathscr{G}_\lambda} = \dot{\bigcup_{\substack{\lambda' \in \Lambda_G^+ \\ \lambda' \leq \lambda}}} \mathscr{G}_{\lambda'} \tag{1}$$

Man stellt fest, dass $\overline{\mathscr{G}_\lambda}$ singulär ist. Es gilt sogar, dass der glatte Ort von $\overline{\mathscr{G}_\lambda}$ nur $\mathscr{G}_\lambda$ ist (siehe [Malkin et al., 2005]).
Wir betrachten im Folgenden einen bestimmten Teil dieser Singularität:
Ein Paar (λ, μ) von dominanten Kogewichten von G mit $\lambda \leq \mu$ heisst *minimale Degeneration von Kogewichten*, falls λ und μ in der Ordnung der dominanten Kogewichte benachbart sind. Mit anderen Worten, falls für alle dominanten Kogewichte ν von G gilt, dass aus $\lambda \leq \nu \leq \mu$ entweder $\nu = \lambda$ oder $\nu = \mu$ folgt.
Das bedeutet nach (1) auch, dass die beiden Orbiten $\mathscr{G}_\mu$ und $\mathscr{G}_\lambda$ benachbart sind im folgenden Sinne: Es gibt keinen Orbit außer $\mathscr{G}_\lambda$, dessen Abschluss $\mathscr{G}_\lambda$ enthält aber nicht $\mathscr{G}_\mu$.
Wir untersuchen also die Singularität, die in $\mathscr{G}_\lambda \cup \mathscr{G}_\mu \subset \overline{\mathscr{G}_\mu}$ auftritt, als Annäherung an $\overline{\mathscr{G}_\mu}$. Diese nennen wir *minimale Degeneration.*
Dazu definieren wir einen transversalen Schnitt

$$\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu} \tag{2}$$

Dabei ist $\mathrm{L}^{<0}G$ eine Untergruppe von $G(k[z^{-1}])$, siehe 2.8, und L_λ ein Punkt von $\mathscr{G}_G$, siehe 2.11.
Das Schema $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ schneidet $\mathscr{G}_\lambda$ nur im Punkt L_λ und hat keinen Schnitt mit $\mathscr{G}_{\lambda'}$ für $\lambda' \ngeq \lambda$, siehe Lemma 5.14.
Der Punkt L_λ ist dann eine isolierte Singularität von $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ und es gilt, dass $\mathscr{G}_\mu \cup \mathscr{G}_\lambda$ glatt äquivalent zu $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ ist (siehe 6.8).

In [Stembridge, 1998] wird eine Charakterisierung aller minimalen Degenerationen von Kogewichten gegeben, siehe Theorem 10.1. Diese umfasst vier Fälle. In zwei dieser Fälle ist es möglich die Singularität in (2) konkret zu berechnen. Wir kommen zu folgendem Resultat:

Theorem 1.1
Sei (λ, μ) eine minimale Degeneration von Kogewichten.

(a) Ist $\mu - \lambda$ eine einfache Kowurzel von G, so gilt

$$(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu} \cong \operatorname{Spec} k[T_1, T_2, T_3]/(T_1^{\lambda_i+2} - T_2T_3)$$

(b) Ist $I_{\lambda,\mu} = J_{\lambda,\mu}$ und $\mu - \lambda = \bar{\alpha}_{\lambda,\mu}$, so gilt

$$(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu} \xrightarrow{\cong} \overline{C_{\min}(\lambda,\mu)}$$

Siehe Abschnitt 10 für die hier verwendete Notation.
Das bedeutet, dass im ersten Fall die Singularität eine Kleinsche Singularität von Typ A ist. Im zweiten Fall ist die Singularität der Abschluss der minimalen nilpotenten Konjugationsklasse $C_{\min}$ in $\operatorname{Lie} M$, wobei M eine Leviuntergruppe von G ist. Diese beiden Arten von Singularitäten nennen wir im Weiteren *minimale Singularitäten.*
Mit diesen Fällen sind alle minimalen Degenerationen von Gruppen erfasst, in denen das Dynkin-Diagramm keine mehrfachen Kanten enthält, insbesondere also Gruppen vom Typ A, D und E. Weiterhin stellen wir fest, dass auch die minimalen Degenerationen im Fall Sp_{2g} damit vollständig behandelt werden (siehe 10.4).

In den beiden anderen Fällen von Theorem 10.1 können auch Singularitäten minimaler Degenerationen auftreten, die keine minimalen Singularitäten sind. Diese heißen *quasi-minimale Singularitäten* und treten in Gruppen auf, deren Dynkin-Diagramme ein Subdiagramm vom Typ C oder G_2 enthalten. In Abschnitt 11 berechnen wir eine Singulatität diesen Typs im Falle der GSp_4.

Der Beweis des Theorems 1.1 hat zwei wesentliche Bestandteile: Zum einen die Reduktion darauf, die affine Grassmannsche der Levi Untergruppe zu betrachten, die zur Teilmenge derjenigen positiven Kowurzeln gehört, die in $\mu - \lambda$ vorkommen. Zum anderen die explizite Berechnung der Singularitäten im Fall $\lambda = 0$ und im Fall $G = \mathrm{PGL}_2$.
Für $\lambda = 0$ wird ein $G(k)$-äquivarianter Isomorphismus konstruiert zwischen einem abgeschlossenen Unterschema von $\mathscr{G}_G$, das den zu μ korrespondierenden Orbit enthält, und dem nilpotenten Kegel der Liealgebra. Dies macht den Zusammenhang zwischen minimalen Degenerationen und Konjugationsklassen deutlich.

In [Juteau, 2008] werden Zerlegungszahlen von perversen Garben auf Singularitäten von minimalen Degenerationen studiert. Diese werden benutzt, um die Vermutung aus [Malkin et al., 2005] zu beweisen, die besagt, dass die quasiminimalen Singularitäten nicht glatt äquivalent zu minimalen Singularitäten sind. Dies Zeigt, dass die obige Fallunterscheidung notwendig ist.

Die Resultate für minimale Singularitäten ähneln dem folgenden Theorem aus [Pappas and Rapoport, 2000, Theorem C] für den Fall $G = \mathrm{GL}_n$:

Theorem 1.2
Jede Schubertvarietät der affinen Grassmannschen der GL_n ist glatt äquivalent zum Abschluss einer nilpotenten Konjugationsklasse der GL_r, für ein passendes r. Insbesondere ist sie normal mit rationalen Singularitäten.

Genauer wird folgender Zusammenhang hergestellt: Sei $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{Z}$ ein dominantes Kogewicht der GL_n. Falls $\lambda_n < 0$ setze $\lambda'_i = \lambda_i - \lambda_n$, sonst $\lambda'_i = \lambda_i$. Dann gilt $\lambda'_1 \geq \cdots \geq \lambda'_n \geq 0$ und $(\lambda'_1, \dots, \lambda'_n)$ ist eine Partition von $r = \sum_i \lambda'_i$. Diese Partition entspricht dem Jordantyp einer nilpotenten $(r \times r)$-Matrix. Der Jordantyp bestimmt damit eine Konjugationsklasse C_λ in $\mathfrak{gl}_r$ (siehe auch [Kraft and Procesi, 1981]).
In [Pappas and Rapoport, 2000] wird gezeigt, dass $\overline{\mathscr{G}_\lambda}$ glatt äquivalent zu $\overline{C_\lambda} \subset \mathfrak{gl}_r$ ist.
Für den Fall, dass λ das minimale Kogewicht ≥ 0 ist, liefert dies einen alternativen Beweis von Theorem 8.3 für den Fall der GL_n: Dann ist $\lambda = (1, 0, \dots, 0, -1)$, $\lambda' = (2, 1, \dots, 1, 0)$ und $r = n$. Außerdem entspricht die Partition $(2, 1, \dots, 1, 0)$ der minimalen (nicht-trivialen) Konjuationsklasse des nilpotenten Kegels in $\mathfrak{gl}_n$.

Alle Resultate der vorliegenden Arbeit sind bereits in [Malkin et al., 2005] enthalten. Es werden Details eingefüllt und elementarere Beweise gegeben, falls möglich. Außerdem stellen wir fest, dass die Bedingung $\mathrm{char}(k) = 0$, die in [Malkin et al., 2005] gestellt wird, nicht notwendig ist.

1 Introduction

We consider the affine Grassmannian $\mathscr{G}_G$ of a closed reductive subgroup G of SL_n, defined over an algebraically closed field k of arbitrary characteristic. $\mathscr{G}_G$ is an ind-representable k-space with k-valued points $G(k(\!(z)\!))/G(k[\![z]\!])$. The construction of $\mathscr{G}_G$ is described in detail in section 3 and section 4.
There is a natural $\mathscr{G}(k[\![z]\!])$-action on $\mathscr{G}_G$. Its orbits are parametrized by the dominant co-weights $\lambda \in \Lambda_G^+$ of G. We denote the orbit corresponding to λ by $\mathscr{G}_\lambda$.
An *(affine) Schubert-variety* in $\mathscr{G}_G$ is the closure $\overline{\mathscr{G}_\lambda} \subset \mathscr{G}_G$ of a $G(k[\![z]\!])$-orbit $\mathscr{G}_\lambda$.
It is known that

$$\overline{\mathscr{G}_\lambda} = \dot{\bigcup_{\substack{\lambda' \in \Lambda_G^+ \\ \lambda' \leq \lambda}}} \mathscr{G}_{\lambda'} \tag{1}$$

One finds that $\overline{\mathscr{G}_\lambda}$ is singular. It is even true that the smooth locus of $\overline{\mathscr{G}_\lambda}$ is only $\mathscr{G}_\lambda$ (see [Malkin et al., 2005]).
In the following we will study a distinguished part of this singularity:
A pair (λ, μ) of dominant co-weights of G with $\lambda \leq \mu$ is called a *minimal degeneration of co-weights*, if λ and μ are neighbours in the order of dominant coweights of G. In other words, if for all dominant co-weights ν of G with $\lambda \leq \nu \leq \mu$ we have either $\nu = \lambda$ or $\nu = \mu$. This means as a consequence of (1) the orbits $\mathscr{G}_\mu$ and $\mathscr{G}_\lambda$ are neighbouring in the following sense: There is no orbit except $\mathscr{G}_\lambda$, whose closure contains $\mathscr{G}_\lambda$ but not $\mathscr{G}_\mu$.
We study the singularity $\mathscr{G}_\lambda \cup \mathscr{G}_\mu \subset \overline{\mathscr{G}_\mu}$ as an approximation to $\overline{\mathscr{G}_\mu}$. It is called *minimal degeneration.*
To this end we define a transverse slice

$$\mathrm{L}^{<0}G \cdot \lambda(z) \cap \overline{\mathscr{G}_\mu} \tag{2}$$

Here $\mathrm{L}^{<0}G$ is a subgroup of $G(k[z^{-1}])$, see 2.8, and L_λ is a point of $\mathscr{G}_G$, see 2.11.
The scheme $\mathrm{L}^{<0}G \cdot \lambda(z) \cap \overline{\mathscr{G}_\mu}$ intersects $\mathscr{G}_\lambda$ only in the point L_λ and has no intersection with $\mathscr{G}_{\lambda'}$ for $\lambda' \not\geq \lambda$, see Lemma 5.14.
Then the point L_λ is an isolated singularity of $\mathrm{L}^{<0}G \cdot \lambda(z) \cap \overline{\mathscr{G}_\mu}$ and $\mathscr{G}_\lambda \cup \mathscr{G}_\mu$ is smoothly equivalent to $\mathrm{L}^{<0}G \cdot \lambda(z) \cap \overline{\mathscr{G}_\mu}$ (see 6.8).

There is a characterization of all minimal degeneration of co-weights given in [Stembridge, 1998], see Theorem 10.1. It contains four cases. In two of these cases it is possible to calculate the singularity in (2) explicitly.
We obtain the following result:

Theorem 1.1
Let (λ, μ) be a minimal degeneration of co-weights.

(a) If $\mu - \lambda$ is a simple co-root of G, then

$$(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu} \cong \operatorname{Spec} k[T_1, T_2, T_3]/(T_1^{\lambda_i+2} - T_2T_3)$$

(b) If $I_{\lambda,\mu} = J_{\lambda,\mu}$ and $\mu - \lambda = \bar{\alpha}_{\lambda,\mu}$, then

$$(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu} \xrightarrow{\cong} \overline{C_{\min}(\lambda, \mu)}$$

See section 10 for the notation. This means in the first case the singularity is a Kleinian singularity of type A. In the second case the singularity is the closure of the minimal nilpotent conjugacy class $C_{\min}$ in $\operatorname{Lie} M$, where M is a Levi subgroup of G. We will call these two types of singularities *minimal singularities.*
These cases cover all simply laced groups, in particular all groups of type A, D, and E. Furthermore we observe that this also deals with all minimal degenerations in the case Sp_{2g} (see 10.4).

In the two other cases of Theorem 10.1 it is possible, that minimal degeneration singularities occur which are not minimal singularities. These are called *quasi-minimal singularities* and occur in groups, whose Dynkin diagram contains a subdiagram of type C or G_2. We calculate a singularity of this type in the case of GSp_4 in section 11.

The proof of Theorem 1.1 result has two central ingredients. The first is the reduction to studying the affine Grassmannian of the Levi subgroup that corresponds to the subsets of simple roots appearing in $\mu - \lambda$. The second is the explicit calculation of singularities in the case of $\lambda = 0$ and in the case of $G = \mathrm{PGL}_2$.
For $\lambda = 0$ we construct a $G(k)$-equivariant isomorphism of a closed subscheme of $\mathscr{G}_G$ containing the orbit corresponding to μ to the nilpotent cone in the Lie algebra. This emphasizes the connection between minimal degenerations and conjugacy classes.

In [Juteau, 2008] the decomposition numbers for perverse sheaves on minimal degeneration singularities are studied. Using these decomposition numbers it is shown that the quasi-minimal singularities are not smoothly equivalent to minimal singularities, as was conjectured in [Malkin et al., 2005]. This shows that the distinction above is necessary.

The results on minimal singularities are similar to a theorem that can be found in [Pappas and Rapoport, 2000, Theorem C] for the case $G = \mathrm{GL}_n$:

Theorem 1.2
Any Schubert variety of affine Grassmannian of GL_n is smoothly equivalent to the closure of a nilpotent conjugacy class for GL_r for suitable r. In particular, it is normal with rational singularities.

More precisely the following connection is established: Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{Z}$ be a dominant co-weight of GL_n. If $\lambda_n < 0$, define $\lambda'_i = \lambda_i - \lambda_n$, else $\lambda'_i = \lambda_i$. Then $\lambda'_1 \geq \cdots \geq \lambda'_n \geq 0$ and $(\lambda'_1, \cdots, \lambda'_n)$ is a partition of $r = \sum_i \lambda'_i$. This partition corresponds to the Jordan type of a nilpotent $(r \times r)$-matrix. This Jordan type determines a conjugacy class C_λ in $\mathfrak{gl}_r$ (see also [Kraft and Procesi, 1981]).
In [Pappas and Rapoport, 2006] it is shown that $\overline{\mathscr{G}_\lambda}$ is smoothly equivalent to $\overline{C_\lambda} \subset \mathfrak{gl}_r$.
In the case that λ is the minimal co-weight ≥ 0, this provides another proof of Theorem 8.3 in the case of GL_n: Then $\lambda = (1, 0, \ldots, 0, -1)$, $\lambda' = (2, 1, \ldots, 1, 0)$ and $r = n$. Furthermore the partition $(2, 1, \ldots, 1, 0)$ corresponds to the minimal (non-trivial) conjugacy class in the nilpotent cone of $\mathfrak{gl}_n$.

All results in the paper at hand can already be found in [Malkin et al., 2005]. However we fill in more details and give more elementary proofs where possible. We also show that it is not necessary to ask for $\mathrm{char}(k) = 0$, as was done in [Malkin et al., 2005].

2 Basic notions

First we will fix notations and recall facts about the affine Grassmannian. For more details we refer to [Beauville and Laszlo, 1994].

Let k be an algebraically closed field of any characteristic and let G be a closed subgroup of SL_n defined over k. By R we denote throughout the text an arbitrary k-algebra. By "scheme" we always mean k-scheme and all morphisms are morphisms over k.

2.1 k-spaces
A priori we work in the category of algebraic k-spaces, meaning functors X from the category of schemes over k to the category of sets that are a sheaf for the fpqc topology.
This is the same as functors from the category of k-algebras to the category of sets such that for any faithfully flat morphism of k-algebras $R \to R'$ the induced diagram

$$X(R) \to X(R') \rightrightarrows X(R' \otimes_R R')$$

is exact (where the double arrow is induced by the two inclusions of R' into the tensor product).
A *k-group* is a group object in the category of k-spaces.

2.2 Ind-schemes
In the category of k-spaces arbitrary inductive limits exist. Details can be found in [Artin, 1962].
Recall that an *ind-scheme* (over k) is a k-space which is the inductive limit in the category of k-spaces of a system of schemes. A functor is called *ind-representable* if it is representable by an ind-scheme.
An ind-scheme X is called reduced resp. irreducible resp. integral if it is the inductive limit of a family X_i of schemes, that are reduced resp. irreducible resp. integral. This does not imply that every family X_i with $\varinjlim_i X_i = X$ is of this type, as one can easily add non-reduced schemes to a system without changing the limit.

2.3 Inductive limits on noetherian Grothendieck-topologies
One can restrict a sheaf on the fpqc-topology to the full subcategory of quasicompact schemes over k. The fpqc-topology restricted to this category is noetherian, meaning every cover can be replaced by a finite subcover (this is obvious).
It is known that on a noetherian Grothendieck-topology that given an inductive system X_i of sheaves, the presheaf $U \mapsto \varinjlim(X_i(U))$ is again a sheaf (and of course the inductive limit in the sense of sheaves). Moreover the functor $\varinjlim$ is exact (see for example [Artin, 1962, Section 5]).

Lemma 2.4
Let X be a quasi-compact scheme and $Y = \varinjlim_i Z_i$ be an ind-scheme, represented by the inductive limit of the schemes Z_i. Then for every morphism $f \colon X \to Y$ there is a i such that f factors through Z_i.

Proof. $Y(X) = \varinjlim_i (Z_i(X))$ by 2.3. But this means that every morphism $R \to Y$ factors through one of the Z_i by definition of the limit. □

2.5
Lemma 2.4 is not a consequence of the universal property of the inductive limit. Recall that the universal property is: giving a morphism *from* $Y = \varinjlim_i Z_i$ into a k-space X is the same as giving a compatible system of morphisms $Z_i \to X$ for all i, whereas Lemma 2.4 talks about morphisms *into* Y.

2.6 Closed and open immersions of k-spaces
A morphism of k-spaces $X \to Y$ is called *representable* if for every k-scheme S and every morphism $S \to Y$ the k-space $X \times_Y S$ is representable.
A morphism $X \to Y$ of k-spaces is called a *closed immersion* if it is representable and for every k-scheme S and every morphism $S \to Y$ the projection $X \times_Y S \to S$ is a closed immersion. We define the properties *open immersion* and *immersion* analogously. With this notions it makes sense to talk of open or closed sub-k-spaces of a given k-space.
It follows from the definition, that these properties are local on Y.

2.7 Closed and open immersions of ind-schemes
Let $f \colon X \to Y$ be a representable morphism of k-spaces with $Y = \varinjlim_i Z_i$ an ind-scheme. Then f is a closed immersion if and only if the basechange $f_{Z_i} \colon X \times_Y Z_i \to Z_i$ is a closed immersion for each i.
One direction is obvious. Assume f_{Z_i} is a closed immersion for each i.
One can test whether f is a closed immersion using only affine test-schemes, i.e. f is a closed immersion if the definition is fulfilled for all affine S: Cover

an arbitrary S by open affine subschemes U_j. If each $X \times_Y U_j$ is representable by a scheme they can be glued together, giving a scheme representing $X \times_Y S$. Testing whether the projection $X \times_Y S \to S$ is a closed immersion of schemes is local on S anyway.
But now the assertion follows from Lemma 2.4: If S is affine then $S \to Y$ factors as $S \to Z_i$ and $X \times_Y S \cong X \times_{Z_i} S$.
The same statement holds for open immersions.
This gives a more intuitive view of sub-ind-schemes. They are open or closed if and only if their intersection with all "finite parts" Z_i is open or closed.

2.8 The loop group
Define

- $\mathrm{L}^{\geq 0}G$ be the k-group $R \mapsto G(R[\![z]\!])$, called the *positive loop group.*
- $\mathrm{L}G$ be the k-group $R \mapsto G(R(\!(z)\!))$, called the *loop group.*
- $\mathrm{L}^{<0}G$ be the k-group $R \mapsto \ker\left(G(R[z^{-1}]) \xrightarrow{z^{-1}\mapsto 0} G(R)\right)$, the kernel of the reduction map.
- $\mathscr{G}_G$ be the k-space $\mathrm{L}G/\,\mathrm{L}^{\geq 0}G$, where the quotient is taken as fpqc-sheaves. This means it is the fpqc-sheaf associated to the functor $R \mapsto G(R(\!(z)\!))/G(R[\![z]\!])$. It is called the *affine Grassmannian* for G.

It is known that in fact all of these objects are actually (ind-)representable. Even more, they are representable as inductive limits of schemes where all transition maps are closed immersions. This special case is called *strict ind-scheme.*

Lemma 2.9
$\mathrm{L}^{\geq 0}G$ is representable by an affine k-scheme.
$\mathrm{L}G$ and $\mathrm{L}^{<0}G$ are ind-representable as limits of affine k-schemes.

Proof. We recall the construction from [Beauville and Laszlo, 1994].
Let M_n be the k-scheme with R-valued points the $(n \times n)$ matrices over R. Let $\mathrm{M}_n^{(N)} = \Pi_{i\geq -N}\,\mathrm{M}_n$. Then $\mathrm{M}_n^{(N)}$ represents the functor that associates to a k-algebra R the set of all matrices of the form $\sum_{i\geq -N} A_i \cdot z^i$ with $A_i \in \mathrm{M}_n(R)$.
Let $\mathrm{L}G^{(N)}(R)$ be the set of all matrices $A \in \mathrm{L}G(R)$ with indeterminate z such that both A and A^{-1} have poles of order $\leq N$. This defines a subfunctor of $\mathrm{L}G$. It is represented by a closed affine subscheme of $\mathrm{M}_n^{(N)}$:
Since SL_n is closed in M_n and G is closed in SL_n, G is a closed subscheme of M_n, defined by a finite set of polynomials $f_i \in k[a_{11}, \ldots, a_{nn}]$. For $N \in \mathbb{N}$ one can find polynomials $P_{i,j}^{(N)} \in k[a_{ij}^{(m)}; i, j = 1, \ldots, n,\ m \geq N]$ such that for $A \in \mathrm{M}_n^{(N)}(R)$ with $A = \sum_{j\geq -N} A_j \cdot z^j$ and $A_j \in \mathrm{M}_n(R)$ we have

$$f_i(A) = \sum_{j\geq -N} P_{i,j}^{(N)}((A_m)_{m\geq -N}) \cdot z^j$$

It is clear that $\mathrm{L}G^{(N)}$ is represented by the closed subscheme of $\mathrm{M}_n^{(N)}$ defined by the vanishing of the $P_{i,j}^{(N)}$. Then $\mathrm{L}^{\geq 0}G = LG^{(0)}$ and $\mathrm{L}G = \varinjlim_N \mathrm{L}G^{(N)}$.

To see that $\mathrm{L}^{<0}G$ is ind-representable consider for $N \in \mathbb{N}$ the k-scheme $\prod_{j=1}^N \mathrm{M}_n$. It represents the functor

$$R \mapsto \{A \in \mathrm{M}_n(R(\!(z)\!)) \mid A = 1_n + \sum_{j=1}^{N} A_j \cdot z^{-j},\ A_j \in \mathrm{M}_n(R)\}$$

(where 1_n is the $n \times n$ identity matrix).
Since each entry of these A is a polynomial in z^{-1} we can define the subfunctor $\mathrm{L}^{<0}_{\geq -N}\mathrm{G}$ of $\prod_{j=1}^N \mathrm{M}_n$ by the vanishing of the polynomials

$$f_i(1_n + \sum_{j=1}^{N} A_j z^{-j}).$$

It is obvious that $\mathrm{L}^{<0}G = \varinjlim_{N\in\mathbb{N}} \mathrm{L}^{<0}_{\geq -N}\mathrm{G}$. $\square$

Define $\mathscr{G}_G^{(N)}$ to be the image of $\mathrm{L}G^{(N)}$ in the quotient $\mathscr{G}_G = \mathrm{L}G/\,\mathrm{L}^{\geq 0}G$. Then $\mathscr{G}_G = \varinjlim_N \mathscr{G}_G^{(N)}$. Our first aim is to prove that the $\mathscr{G}_G^{(N)}$ are representable. Then it is clear that the directed system is given by closed immersions.

2.10 Notations
Let G be reductive and connected. This assumption will not be used until section 5 and is not needed for $\mathscr{G}_G$ to be representable. But we will use the following notation for SL_n, so we introduce it now.
Inside SL_n we have the standard torus and Borel subgroup. Every Borel subgroup B of G is of the form $B = (B' \cap G)^\circ$, where B' is a Borel subgroup of SL_n and $(\cdot)^\circ$ denotes the connected component of the unity. By conjugating SL_n we can assume B' to be the standard Borel. This changes the embedding of G in SL_n, which means that in the case of classical groups we may not end up with the natural embedding. But we only do explicit calculations in the cases of Sp_{2g} and PGL_2 and there is no problem in this case.
This means we have $T \subset B \subset G$, a torus and Borel subgroup of G, lying inside the standard torus and Borel subgroup of SL_n.
Let I_G be the set of vertices in the Dynkin diagram associated to G. Let $\check{\alpha}_i$, $i \in I_G$ be the simple co-roots and Q_G the co-root lattice. Let $\check{\omega}_i$, $i \in I_G$ be the fundamental co-weights, Λ_G the co-weight lattice and Λ_G^+ the dominant co-weights.
By construction the co-weight lattice of G is contained in the co-weight lattice of SL_n and $\Lambda_G^+ \subset \Lambda_{\mathrm{SL}_n}^+$.

2.11 The point of $\mathscr{G}_G$ assiociated to $\lambda \in \Lambda_G$

One can also view Λ_G as a subgroup of $T(k(\!(z)\!))$ by sending λ to $\lambda(z)$. Denote the image of $\lambda(z) \in T(k(\!(z)\!))$ in $\mathscr{G}_G$ by L_λ. The assignment $\lambda \mapsto L_\lambda$ is a bijection. This can be seen choosing an isomorphism $T \cong \mathbb{G}_m^r$. Then

$$\begin{aligned} \Lambda_G &= \mathrm{Hom}(\mathbb{G}_m, T) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m^r) \\ &= (k(\!(z)\!)/k[\![z]\!])^r = (\mathrm{LT} \,/\, \mathrm{L}^{\geq 0}\,\mathrm{T})(k) \end{aligned} \tag{3}$$

In the case of SL_n with the standard torus the statement above can be reformulated in the following way: Given $\lambda \in \Lambda_G^+$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ we write z^λ for $\mathrm{diag}(z^{\lambda_1}, \ldots, z^{\lambda_n}) = \lambda(z)$ as a point of $\mathrm{L}G(k)$

Since we chose the torus T to be contained in the diagonal matrices in SL_n, $\lambda(z)$ is of course a diagonal matrix where all entries are monomials in z. But since $\lambda(z)$ depends on the embedding of G into SL_n we use the more suggestive notation z^λ only when it is clear which embedding we refer to.

3 The affine Grassmannian for SL_n

In this section we recall the proof of [Beauville and Laszlo, 1994], showing that $\mathscr{G}_{\mathrm{SL}_n}$ is ind-representable. Let L_0 be the $R[\![z]\!]$-module $R[\![z]\!]^n$. Recall the following concept:

3.1 Lattices in $R(\!(z)\!)^n$

A *lattice in* $R(\!(z)\!)^n$ is a $R[\![z]\!]$-submodule L of $R(\!(z)\!)^n$ that is projective and of rank n such that

$$L \otimes_{R[\![z]\!]} R(\!(z)\!) = R(\!(z)\!)^n.$$

A lattice L in $R(\!(z)\!)^n$ is called *special* if $\bigwedge^n L = R[\![z]\!]$ as a $R[\![z]\!]$-submodule of $R(\!(z)\!) \cong \bigwedge^n R(\!(z)\!)^n$.

3.2

We will need the following fact from commutative algebra:

Let R_i, $i \in \mathbb{N}$ be an inverse system of rings with surjective connection maps $R_i \twoheadrightarrow R_{i-1}$ and limit $\hat{R}$. Let M be a finitely generated $\hat{R}$-submodule of $(\hat{R})^n$. If all $M_i := M \otimes_{\hat{R}} R_i$ are projective R_i-modules, then M is a projective $\hat{R}$-module. The proof is straightforward.

Lemma 3.3

Let $L \subset R(\!(z)\!)^n$ be a finitely generated $R[\![z]\!]$-module. Then the following are equivalent:

(*a*) $L \otimes_{R[\![z]\!]} R(\!(z)\!) = R(\!(z)\!)^n$

(b) there is a $N \in \mathbb{N}$ such that $z^N L_0 \subset L \subset z^{-N} L_0$

Proof. Observe that $L \otimes_{R[\![z]\!]} R(\!(z)\!) = \bigcup_{i \in \mathbb{N}} z^{-i} \cdot L$.
Let L satisfy a). Since it is a finitely generated R-module, there is $N \in \mathbb{N}$ such that $z^N L \subset L_0$.
Because of $\bigcup_{i \in \mathbb{N}} z^{-i} \cdot L = R(\!(z)\!)^n$ there is $i \in \mathbb{N}$ such that $z^{-i} L$ contains the standard basis of $R(\!(z)\!)^n$, showing $z^{-i} L \supset L_0$.
The other implication is obvious. □

Lemma 3.4
Let L satisfy the conditions of Lemma 3.3 and fix $N \in \mathbb{N}$ with

$$z^N L_0 \subset L \subset z^{-N} L_0$$

The following are equivalent:

(a) There is $i \in \mathbb{N}$, $i \geq 2N$ such that $L/z^i L$ is a projective R-module.

(b) $z^{-N} L_0 / L$ is a projective R-module.

(c) $z^{-N} L_0 / L$ and $L / z^N L_0$ are projective R-modules.

(d) $L/z^i L$ is a projective R-module for every $i \in \mathbb{N}$.

Proof.

$a) \Rightarrow b)$ There is an exact sequence of R-modules

$$0 \to z^N L_0 / z^i L \to L / z^i L \to L / z^N L_0 \to 0$$

The natural projection of R-modules $z^{-N} L_0 / z^i L \twoheadrightarrow z^N L_0 / z^i L$ induces a retraction $L/z^i L \twoheadrightarrow z^N L_0 / z^i L$. Therefore the sequence splits and $z^{-N} L_0 / L$ is projective.

$b) \Rightarrow c)$ Assume b). Then the exact sequence

$$0 \to L / z^N L_0 \to z^{-N} L_0 / z^N L_0 \to z^{-N} L_0 / L \to 0$$

splits, showing c) since the middle term is free.

$c) \Rightarrow d)$ First let $i = 2N$.
If $z^{-N} L_0 / L$ and $L / z^N L_0$ are projective, then the exact sequence

$$0 \to z^N L_0 / z^i L \to L / z^i L \to L / z^N L_0 \to 0$$

splits (as in "$a) \Rightarrow b)$") and we conclude $L / z^{2N} L$ is projective over R.

Using c) the exact sequence

$$0 \to L / z^{N+1} L_0 \to z^{-N} L_0 / z^{N+1} L_0 \to z^{-N} L_0 / L \to 0$$

splits. Since $z^{-N}L_0/z^{N+1}L_0$ is free, $L/z^{N+1}L_0$ is projective, too. Consequently

$$0 \to z^{N+1}L_0/z^{2N+1}L \to L/z^{2N+1}L \to L/z^{N+1}L_0 \to 0$$

splits. As $z^{N+1}L_0/z^{2N+1}L \cong z^{-N}L_0/L$ is projective by assumption, the middle term $L/z^{2N+1}L$ is projective, too. This is the case $i = 2N+1$. Now with the argument of "$a) \Rightarrow b)$" applied to the exact sequence

$$0 \to z^{N}L_0/z^{2N+1}L \to L/z^{2N+1}L \to L/z^{N}L_0 \to 0$$

we find that $z^N L_0/z^{2N+1}L \cong z^{-N-1}L_0/L$ is projective. This is $b)$ for $N+1$. Obviously $z^{N+1}L_0 \subset L \subset z^{-N-1}L_0$. Hence we can use induction on N to conclude $d)$ for all $i \geq 2N$.

For $i < 2N$ the assertion follows from the exact sequence

$$0 \to z^i L/z^{2N+i}L \to L/z^{2N+i}L \to L/z^i L \to 0$$

since the middle and left term are projective.

$d) \Rightarrow a)$ is obvious.

□

Lemma 3.5
Let $L \subset R(\!(z)\!)^n$ be a finitely generated $R[\![z]\!]$-module satisfying the conditions in Lemma 3.3. Then the following are equivalent:

(a) L is a projective $R[\![z]\!]$-module.

(b) L is locally on R a free $R[\![z]\!]$-module. By this we mean there are $a_1, \ldots, a_r \in R$ generating the unit ideal in R such that

$$L \otimes_{R[\![z]\!]} R_{a_i}[\![z]\!]$$

is a free $R_{a_i}[\![z]\!]$-module.

(c) $z^{-N}L_0/L$ is a projective R-module.

Proof.

$a) \Rightarrow b)$ If L is a projective $R[\![z]\!]$-module then there exist $f_1, \ldots, f_r \in R[\![z]\!]$ generating the unit ideal in $R[\![z]\!]$ such that $L \otimes_{R[\![z]\!]} R[\![z]\!]_{f_i}$ is free for all i. Let a_i be the coefficient of f_i in degree 0. Then $a_1, \ldots, a_r$ generate the unit ideal in R. Also $R[\![z]\!] \to R[\![z]\!]_{f_i}$ factors through $R_{a_i}[\![z]\!]$. This means $L \otimes_{R[\![z]\!]} R_{a_i}[\![z]\!] = L \otimes_{R[\![z]\!]} R[\![z]\!]_{f_i} \otimes_{R[\![z]\!]_{f_i}} R_{a_i}[\![z]\!]$ is free.

$b) \Rightarrow c)$ We claim that $c)$ is local on R. In other words if $a_1, \ldots, a_r \in R$ generate the unit ideal in R and $z^{-N}R^n_{a_i}/\left(L \otimes_{R[\![z]\!]} R_{a_i}[\![z]\!]\right)$ is a projective

R_{a_i}-module, then $z^{-N}L_0/L$ is a projective R-module.
Indeed, consider the commutative diagram

$$\begin{tikzcd}
\left(L/z^N L_0\right) \otimes_R R_{a_i} \arrow[r, hook] \arrow[d, "\varphi'"] & \left(z^{-N}L_0/z^N L_0\right) \otimes_R R_{a_i} \arrow[r, two heads] \arrow[d, "\varphi"] & \left(z^{-N}L_0/L\right) \otimes_R R_{a_i} \arrow[d, "\varphi''"] \\
\left(L \otimes_{R[\![z]\!]} R_{a_i}[\![z]\!]\right)/z^N R_{a_i}[\![z]\!]^n \arrow[r, hook] & z^{-N}R_{a_i}[\![z]\!]^n/z^N R_{a_i}[\![z]\!] \arrow[r, two heads] & z^{-N}R_{a_i}[\![z]\!]^n/\left(L \otimes_{R[\![z]\!]} R_{a_i}[\![z]\!]\right)
\end{tikzcd}$$

with exact rows.
Here φ is obviously an isomorphism. But so is φ' since

$$L/z^N L_0 = L \otimes_{R[\![z]\!]} R[z]/z^N$$

By the 5-lemma, φ'' is an isomorphism. This proves the claim.
Using the claim we may assume that L is a free $R[\![z]\!]$ module. But then $L/z^{2N}L$ is a free R-module, in particular projective. Using Lemma 3.4 we conclude $c)$.

$c) \Rightarrow a)$ By Lemma 3.4 $L/z^i L$ is projective over R for all $i \in \mathbb{N}$. In particular L/zL is projective over R. Inductively we can split $L/z^j L$ for $j \in \mathbb{N}$ into a direct sum

$$L/z^j L \cong \bigoplus_{i=1}^{j} (L/zL)$$

This isomorphism of R-modules is even an isomorphism of $R[z]/z^j$-modules if the z-action on the right is defined by shifting to the next summand.
Choose an epimorphism $E \to L$ with E a free $R[\![z]\!]$-module and decompose

$$E/z^n E = \bigoplus_{i=1}^{j} (E/zE)$$

The natural $R[z]/z^j$-epimorphism $\bigoplus_{i=1}^{j}(G/zG) \to \bigoplus_{i=1}^{j}(L/zL)$ has a splitting, induced by the R-linear splitting on every summand. This proves $L/z^j L$ is projective over $R[z]/z^j$. Using 3.2 we find that L is projective over $R[\![z]\!]$.

$\square$

Lemma 3.6
Fix $N \in \mathbb{N}$. The functor $\mathcal{Q}^N$ associating to a k-algebra R the set of special lattices L in $R(\!(z)\!)^n$ such that

$$z^N L_0 \subset L \subset z^{-N} L_0$$

is representable.

Proof. We repeat the construction of [Beauville and Laszlo, 1994]:

Let $\mathrm{Grass}^z(nN, 2nN)$ be the Grassmannian parametrizing z-stable subspaces of a free module F_N of rank n over $k[z]/(z^{2N})$ (F_N has dimension $2nN$ as a vectorspace over k).

Then we claim that $\mathcal{Q}^N$ is represented by a closed subscheme of $\mathrm{Grass}^z(nN, 2nN)$ with the same underlying topological space.
This correspondence is obtained by sending a lattice $L \in \mathcal{Q}^N(R)$ to its image $\bar{L} = L/(z^r \cdot R[\![z]\!]^n)$ in the quotient

$$(z^{-r} \cdot R[\![z]\!]^n)/(z^r \cdot R[\![z]\!]^n) \cong F_N \otimes_k R.$$

If R is a field, then L is special if and only if $\bar{L}$ has dimension nN: by the elementary divisor theorem there is a $R[\![z]\!]$-basis $e_1, \ldots, e_r$ of $R[\![z]\!]^n$ such that the $R[\![z]\!]$-module L has a basis of the form $(z^{d_1}e_1, \ldots, z^{d_r}e_r)$ with $-N \leq d_i \leq N$. Both conditions are then equivalent to $\sum d_i = 0$.
For general R there is an exact sequence

$$0 \to \bar{L} \to F \to F/\bar{L} \to 0$$

of R-modules. It splits since $F/\bar{L}$ is projective by Lemma 3.5. This means $\bar{L}$ is a direct summand of F. Given any morphism $R \to K$ into a field K we know $\dim_K \left(K \otimes_R \bar{L}\right) = nN$. Hence $\bar{L}$ is of rank nN.
Conversely, let $\operatorname{rank} \bar{L} = nN$. Then locally over $\mathrm{Spec}(R)$ one has $\bigwedge^r \bar{L} = z^{-rN} \cdot x \cdot R[\![z]\!]$ for suitable $x \in R[\![z]\!]$, since $L \subset z^{-N}R[\![z]\!]^n$. By the case over a field we know that for each homomorphism $R \to K$ into a field, the image of x in $K[\![z]\!]$ is of the form $z^{nN}u$ where u is a unit of $K[\![z]\!]$, since $K \otimes_R \bar{L}$ is special. Therefore the coefficients of $z^0, \ldots, z^{nN-1}$ in x have to be nilpotent whereas the coefficient of z^{nN} is a unit. Let I_L be the nilpotent ideal generated by the coefficients of $z^0, \ldots, z^{nN-1}$ in x.
Given an arbitrary ring homomorphism $u\colon R \to R'$, the lattice

$$R' \otimes_R L \subset R'(\!(z)\!)$$

is special if and only if $u(I_L) = 0$. This means the functor associating to R the set of direct z-stable sub-R-modules of $R \otimes_k F_N$ of rank nN, such that the corresponding lattice is special is represented by a closed subscheme of $\mathrm{Grass}^z(nN, 2nN)$, defined by a nilpotent ideal.
We constructed an isomorphism of functors between this functor and $\mathcal{Q}^N$. □

3.7 A basis of F_N
There is a basis of F_N given by the images of $z^j \cdot e_k$, where $k = 1, \ldots, n$, $j = -N, \ldots, N-1$. Write $b_{2N(j+N)+k} := z^j \cdot e_k$, ordering the basis is first by the z-power j, then by the index k. This is a useful notation for writing down matrices.

3.8 The lattice functor is ind-representable

As a consequence of 3.6 we know that the functor associating to R the set of special lattices in $R(\!(z)\!)^n$ is ind-representable, since it is the inductive limit of the $\mathcal{Q}^N$ (using Lemma 3.3). In particular, it is a k-space.

Lemma 3.9

$\mathscr{G}_{\mathrm{SL}_n}$ is the functor associating to R the set of special lattices in $R(\!(z)\!)^n$ and $\mathscr{G}_{\mathrm{SL}_n}^{(N)}$ its subfunctor $\mathcal{Q}^N$. In particular, $\mathscr{G}_{\mathrm{SL}_n}$ is ind-representable.

Proof. Observe that the functor $R \mapsto \mathrm{SL}_n(R(\!(z)\!))/\mathrm{SL}_n(R[\![z]\!])$ parametrizes free $R[\![z]\!]$-submodules W of $R(\!(z)\!)^n$ of rank n such that $z^N L_0 \subset W \subset z^{-N} L_0$ for suitable $N \in \mathbb{N}$ and $\bigwedge^n W = R[\![z]\!]$.

Indeed, let $\bar{g} \in \mathrm{SL}_n(R(\!(z)\!))/\mathrm{SL}_n(R[\![z]\!])$ and let $g \in \mathrm{SL}_n(R(\!(z)\!))$ be a representative of $\bar{g}$. Let W be the $R[\![z]\!]$-submodule of $R(\!(z)\!)^n$ generated by the columns of g. W is obviously independent of the choice of g. Since g is invertible, W is a free $R[\![z]\!]$-module of rank n. Hence there exists $N \in \mathbb{N}$ with $z^N \cdot W \subset R[\![z]\!]^n$. Let N' be the maximal z^{-1}-power in g^{-1}. Then $z^{-N'} \cdot W \supset R[\![z]\!]^n$. It is obvious that $\bigwedge^n W = R[\![z]\!]$, since $\det(g) = 1$.

Using Lemma 3.5 we know that Zariski-locally on R, every lattice is a free $R[\![z]\!]$-module. Being special and the condition

$$z^N L_0 \subset W \subset z^{-N} L_0$$

are obviously stable under tensoring. This means for every lattice L in $R(\!(z)\!)^n$ there is a faithfully flat morphism $R \to R'$ such that $L \otimes_R R'$ is a free $R'[\![z]\!]$-module of the type parametrized by $\mathrm{SL}_n(R'(\!(z)\!))/\mathrm{SL}_n(R'[\![z]\!])$. As we know that the lattice functor is a fpqc-sheaf, this means it is the fpqc-sheafification of the functor $R \mapsto \mathrm{SL}_n(R(\!(z)\!))/\mathrm{SL}_n(R[\![z]\!])$.

The statement about $\mathscr{G}_{\mathrm{SL}_n}^{(N)}$ follows immediately. $\square$

3.10 $\mathscr{G}_G$ is not reduced in general

Consider $\mathscr{G}_{\mathrm{GL}_1}$. Dropping the specialty condition in Lemma 3.9 one can show that $\mathscr{G}_{\mathrm{GL}_n}$ represents the functor $R \mapsto \{\text{ lattices in } R(\!(z)\!)^n\}$. Since every lattice in $k(\!(z)\!)$ is free, we have $\mathscr{G}_{\mathrm{GL}_1}(k) = \mathrm{GL}(k(\!(z)\!))/\mathrm{GL}(k[\![z]\!])$ by the proof of Lemma 3.9. But $\mathrm{GL}(k(\!(z)\!))/\mathrm{GL}(k[\![z]\!])$ is just countable many points, given by the representatives z^i, $i \in \mathbb{Z}$ and therefore

$$\mathscr{G}_{\mathrm{GL}_1\,\mathrm{red}} = \coprod_{\mathbb{Z}} \operatorname{Spec} k.$$

If $\mathscr{G}_{\mathrm{GL}_1}$ was reduced, every $k[\varepsilon]/\varepsilon^2$-valued point would factor over one of the copies of $\operatorname{Spec} k$. But there are more points in $\mathscr{G}_{\mathrm{GL}_1}(k[\varepsilon]/\varepsilon^2)$, for example the one given by $\varepsilon + z$. Hence $\mathscr{G}_G$ is not reduced in general, even if G is.

Now we take a closer look at the structure of $\mathscr{G}_{\mathrm{SL}_n}$:

Lemma 3.11 ([Faltings, 2003, Lemma 2])
The action of $\mathrm{L}^{<0}\mathrm{SL}$ on the element L_0 of $\mathscr{G}_{\mathrm{SL}_n}$ corresponding to the standard lattice $k[\![z]\!]^n$ defines an isomorphism that is given on the R-valued points as

$$\mathrm{L}^{<0}\mathrm{SL}(R) \xrightarrow{\cong} \{L \in \mathscr{G}_{\mathrm{SL}_n} \mid L \oplus_R z^{-1}R[z^{-1}]^n = R(\!(z)\!)^n\}$$

Proof. We reproduce the proof.
Since the elements of $\mathrm{L}^{<0}\mathrm{SL}$ are of the form $1_n + z^{-1}M$ with $M \in \mathrm{M}_n(R[z^{-1}])$ it is clear that for $h \in \mathrm{L}^{<0}G$ the lattice $h \cdot R[\![z]\!]^n$ satisfies the equation

$$h \cdot R[\![z]\!]^n \oplus_R z^{-1}R[z^{-1}]^n = R(\!(z)\!)^n.$$

Let $L \oplus z^{-1}R[z^{-1}]^n = R(\!(z)\!)^n$. We claim that there is a unique $h \in \mathrm{M}_n(R(\!(z)\!))$ of the above form such that all columns of h are in L. In what follows all vectors are column vectors and the transpose of a vector v is denoted by v^t.
Let e_i, $i = 1, \ldots, n$ be the standard basis of $R(\!(z)\!)^n$. By assumption there are unique $v_i \in L$ and $w_i \in z^{-1}R[z^{-1}]^n$ such that $v_i + w_i = e_i$.
Let $h = 1_n - \sum_{i=1}^n e_i^t \cdot w_i$. Then $h \cdot e_i = e_i - w_i = v_i \in L$ and the uniqueness is evident.
Now we claim that $L = h \cdot R[\![z]\!]^n$, i.e. the columns v_i of h generate L.
Let $r \gg 0$ such that $z^r \cdot h \in M_N(R[z])$ and $z^r \cdot R[\![z]\!]^n \subset z \cdot L$. To show the claim it suffices to prove that the images $\bar{v}_i$ of the v_i generate $L/(z^r \cdot R[\![z]\!]^n)$ as an $R[\![z]\!]$-module:
Let $x \in L$. If the $\bar{v}_i$ generate $L/(z^r \cdot R[\![z]\!]^n)$ we can write $x = \sum_{i=1}^n x_i v_i + y_0$ with $x_i \in R[\![z]\!]$ and $y_0 \in z^r \cdot R[\![z]\!]^n$.
This means there is $y_0 \in z^r \cdot R[\![z]\!]^n \subset z \cdot L$ with

$$x \equiv y_0 \mod \mathrm{Span}_{R[\![z]\!]}(v_1, \ldots, v_n).$$

Iterating the process (with y_0 as x) we get $y_k \in z^k \cdot L$ with

$$x \equiv y_k \mod \mathrm{Span}_{R[\![z]\!]}(v_1, \ldots, v_n)$$

for all $k \in \mathbb{N}$. This means the sequence y_k converges to 0 and therefore

$$L = \mathrm{Span}_{R[\![z]\!]}(v_1, \ldots, v_n).$$

Now we show that the $\bar{v}_i$ generate $L/(z^r \cdot R[\![z]\!]^n)$ as an $R[\![z]\!]$-module. Let $\bar{x} \in L/(z^r \cdot R[\![z]\!]^n)$ and let

$$\mathrm{pr}\colon L \subset R(\!(z)\!)^n = R[\![z]\!]^n \oplus_R z^{-1}R[z^{-1}]^n \to R[\![z]\!]^n$$

be the first projection.
There is a $x \in L$ representing $\bar{x}$ such that $\mathrm{pr}(x)$ is a linear combination of $z^j \cdot e_i$ with $j \leq r$. The projection $\mathrm{pr}(h \cdot z^j e_i)$ is of the form "$z^j e_i$+ lower z-terms" because of the form of h.
In particular $\mathrm{pr}(h \cdot e_i) = e_i$. By induction

$$\mathrm{pr}(h \cdot z^j e_i) \equiv z^j e_i \mod \mathrm{Span}_{R[\![z]\!]}(he_1, \ldots he_n).$$

And therefore

$$z^j e_i \in \mathrm{Span}_{R[\![z]\!]}(he_1, \ldots he_n) = \mathrm{Span}_{R[\![z]\!]}(v_1, \ldots, v_n).$$

This means $\mathrm{pr}(x) \in \mathrm{Span}_{R[\![z]\!]}(v_1, \dots, v_n) \subset L$. But

$$x - \mathrm{pr}(x) \in \left(L \cap z^{-1}R[z^{-1}]^n\right) = (0)$$

so $x = \mathrm{pr}(x) \in \mathrm{Span}_{R[\![z]\!]}(v_1, \dots, v_n)$. This proves the claim.
To see that h is actually an element of $\mathrm{L}^{<0}G(R)$, we observe that L and $R[\![z]\!]^n$ have the same $R[\![z]\!]$-determinant (L is special). Therefore the determinant of h is a unit in $R[\![z]\!]$, in particular it contains no negative z-powers and thus it is 1. $\square$

3.12 A cover of the Grassmannian
We recall some standard facts about the Grassmannian $\mathrm{Grass}(r,s)$ parametrizing rank r submodules of a free module of rank s: There is an open cover of $\mathrm{Grass}(r,s)$ given on the k-valued points as

$$U_J = \{V \subset k^s \mid V \oplus \mathrm{Span}(f_j; j \notin J) = k^s\}$$

where J is a subset of $\{1, \dots, s\}$ with r elements and f_j is a basis of k^s.
This can be reformulated using another description of $\mathrm{Grass}(r,s)$: The k-valued point of the Grassmannian are given by $\mathrm{M}_{(s,r)}(k)/\mathrm{GL}_r(k)$, where $\mathrm{M}_{(s,r)}(k)$ is the set of $(s \times r)$-matrices with rank r and entries in k and the operation is given by matrix multiplication. The correspondence is given by sending a matrix to the subspace of k^s generated by the columns of the matrix. This is well-defined on a co-set since the right action by $\mathrm{GL}_r(k)$ changes the columns only by invertible linear combination.
Then a subspace V is in U_J if and only if in a matrix representation of V in the above sense the submatrix consisting of the columns whose index is in J is invertible. This condition is independent of the representative we choose since it is stable under the $\mathrm{GL}_r(k)$-action. With this description we see that $U_J \cong \mathbb{A}^{r(s-r)}$ by choosing the representative where the submatrix consisting of the columns whose index is in J is the identity matrix 1_r.

3.13
Lemma 3.11 motivates the following definition:
Let $\lambda \in \Lambda^+_{\mathrm{SL}_n}$, $\lambda = (\lambda_1, \dots, \lambda_n)$. Then L_λ (defined in 2.11) is generated by the columns of the diagonal matrix z^λ. This means $(z^{\lambda_1}e_1, \dots, z^{\lambda_n}e_n)$ is a set of generators of L_λ as a $k[\![z]\!]$-module. For $N \geq |\lambda_i|$, $i = 1, \dots, n$ we have $L_\lambda \in \mathscr{G}^{(N)}_{\mathrm{SL}_n}$.
Using the construction in the proof of Lemma 3.6 we interpret L_λ as a point of $\mathrm{Grass}^z(nN, 2nN)$. Then the image in $\mathrm{Grass}^z(nN, 2nN)$ (which we also denote by L_λ) is a k-vectorspace with basis $z^j e_i$, $j \geq \lambda_i$.
Let L^-_λ be the k-vectorspace with basis $z^j e_i$, $i = 1, \dots, n$ with $j < \lambda_i$. Then we have $L_\lambda \oplus L^-_\lambda = k(\!(z)\!)^n$.

On R-valued points we define $\pi_\lambda \colon R((z))^n = L_\lambda \oplus L_\lambda^- \to L_\lambda$ to be the first projection. Then we define $\mathcal{U}_\lambda$ to be

$$\mathcal{U}_\lambda := \{L \in \mathscr{G}_{\mathrm{SL}_n} \mid \text{ the projection } \pi_\lambda|_L \colon L \to L_\lambda \text{ is an isomorphism } \}.$$

For the case $\lambda = 0$ this is just the $\mathrm{L}^{<0}\mathrm{SL}$-orbit of $R[\![z]\!]^n$ by Lemma 3.11. Let $\mathcal{U}_\lambda^{(N)} := \mathcal{U}_\lambda \cap \mathscr{G}_{\mathrm{SL}_n}^{(N)}$.
Then $\mathcal{U}_\lambda^{(N)}$ is one of the open subsets of the cover of $\mathscr{G}_{\mathrm{SL}_n}^{(N)}$ induced by the cover of the Grassmannian described in 3.12:
$\mathcal{U}_\lambda^{(N)} \cong U_J \cap \mathrm{Grass}^z(nN, 2nN)$ where $J = \{j \in \{1, \dots, n\} \mid b_j \in L_\lambda\}$ and b_j as in 3.7. Let $U_\lambda := U_J$, in particular

$$U_0 = \{V \subset k^{2nN} \mid V \oplus \mathrm{Span}(z^k e_i; i = 1, \dots, n,\ k = -1, \dots, -N) = k^{2nN}\}.$$

This means that the $\mathcal{U}_\lambda$ are open in $\mathscr{G}_{\mathrm{SL}_n}$. It is easy to see that $\mathcal{U}_\lambda$ is stable under the action of $\mathrm{L}^{<0}\mathrm{SL}$. Therefore $\mathcal{U}_\lambda^{(N)}$ contains $\mathrm{L}^{<0}\mathrm{SL} \cdot z^\lambda \cap \mathscr{G}_{\mathrm{SL}_n}^{(N)}$, a fact which we will need later.

By Lemma 3.11 we obtain for the case $\lambda = 0$:

Lemma 3.14
$\mathrm{L}^{<0}\mathrm{SL} \cdot L_0$ is an open orbit of the $\mathrm{L}^{<0}\mathrm{SL}$-action on $\mathscr{G}_{\mathrm{SL}_n}$

$$\mathrm{L}^{<0}\mathrm{SL} \cdot L_0 \cap \mathscr{G}_{\mathrm{SL}_n}^{(N)} = \mathcal{U}_0^{(N)} \xrightarrow{\cong} U_0 \cap \mathrm{Grass}^z(nN, 2nN)$$

where U_0 as above.

3.15 The open cell in $\mathscr{G}_G$
$\mathrm{L}^{<0}\mathrm{SL} \cdot L_0$ is called the *open cell* of $\mathscr{G}_{\mathrm{SL}_n}$.
For arbitrary λ the statement of Lemma 3.14 is no longer true. In general $\mathrm{L}^{<0}G \cdot L_\lambda$ is locally closed in $\mathscr{G}_G$ (Lemma 6.7). In fact we will see that it is closed in $\mathcal{U}_\lambda$ (this follows from Lemma 5.13, a)).

4 The affine Grassmannian for general G

Let G be a closed reductive subgroup of SL_n.

4.1
By Lemma 3.11 and using the transitive LSL-action on $\mathscr{G}_{\mathrm{SL}_n}$ we can see that $\mathscr{G}_{\mathrm{SL}_n}$ has an open cover of k-spaces isomorphic to $\mathrm{L}^{<0}\mathrm{SL}$, with isomorphisms given by the $\mathrm{L}^{<0}\mathrm{SL}$-action on $\mathscr{G}_{\mathrm{SL}_n}$. This means the quotient map $\mathrm{LSL} \to \mathscr{G}_{\mathrm{SL}_n}$ is locally a trivial fibration with fibre $\mathrm{L}^{\geq 0}\mathrm{SL}$. We want to extend this result to $\mathscr{G}_G$.

Lemma 4.2 ([Faltings, 2003, Corollary 3])
The multiplication morphism $\mathfrak{m}\colon \mathrm{L}^{<0}G \times \mathrm{L}^{\geq 0}G \to \mathrm{L}G$ is an open immersion.

Proof. We recall the proof:
One can find a representation $\varrho\colon \mathrm{SL}_n \to \mathrm{GL}_d$ (defined over k) such that G is the stabilizer of an element $e \in \mathbb{P}^d_k$ (this is always possible, see [Springer, 1998, Chapter 5.5]).
The lemma is true for $G = \mathrm{SL}_n$ by 4.1.
Thus we only need to prove that the image of $\mathrm{L}^{<0}G \times \mathrm{L}^{\geq 0}G$ in $\mathrm{L}G$ is the image of

$$\mathfrak{m}\colon \mathrm{L}^{<0}\mathrm{SL} \times \mathrm{L}^{\geq 0}\mathrm{SL} \to \mathrm{LSL}$$

intersected with $\mathrm{L}G$. In other words we have to show that if $g = g_- \cdot g_+ \in \mathrm{L}G(R)$ with $g_- \in \mathrm{L}^{<0}\mathrm{SL}(R)$ and $g_+ \in \mathrm{L}^{\geq 0}\mathrm{SL}(R)$ then g_- and g_+ are already in $\mathrm{L}G(R)$. But

$$\varrho(g_-) \cdot \varrho(g_+) \cdot e = \varrho(g) \cdot e = e$$

and thus

$$\varrho(g_-^{-1}) \cdot e = \varrho(g_+) \cdot e \in R(\!(z)\!)^h.$$

But then $\varrho(g_-^{-1}) \cdot e = \varrho(g_+) \cdot e$ lies in $\mathbb{P}^d_k(R[\![z]\!])$, since $g_+ \in \mathrm{L}^{\geq 0}\mathrm{SL}(R)$. It also lies in $\mathbb{P}^d_k(R[z^{-1}])$ such that its image under $\mathbb{P}^d_k(R[z^{-1}]) \to \mathbb{P}^d_k(R)$, $z^{-1} \mapsto 0$ is e since $g_- \in \mathrm{L}^{<0}\mathrm{SL}$. Hence $\varrho(g_-^{-1}) \cdot e = \varrho(g_+) \cdot e = e$. $\square$

4.3
Define $\mathcal{U}_{0,G} := \mathcal{U}_0 \cap \mathscr{G}_G$ and $\mathcal{U}^{(N)}_{0,G} := \mathcal{U}^{(N)}_0 \cap \mathscr{G}_G$. It is obvious that $\mathcal{U}_{0,G}$ and its $\mathrm{L}G$-translates are an open cover of $\mathscr{G}_G$. Using the above lemma we get an open cover of $\mathrm{L}G$ such that each open set is isomorphic to $\mathrm{L}^{<0}G \times \mathrm{L}^{\geq 0}G$ via $\mathfrak{m}$. This means that the quotient map $\mathrm{L}G \to \mathscr{G}_G$ is locally the first projection of the product $\mathrm{L}^{<0}G \times \mathrm{L}^{\geq 0}G$, i.e. a trivial fibration. In particular:

$$\begin{aligned} \mathrm{L}^{<0}G &\xrightarrow{\cong} \mathcal{U}_{0,G} \\ g &\mapsto g \cdot L_0 \end{aligned}$$

Theorem 4.4
The map $f\colon \mathscr{G}_G \to \mathscr{G}_{\mathrm{SL}_n}$ is a closed immersion of k-spaces. In particular $\mathscr{G}_G$ is an ind-scheme.

Proof. The question is local on $\mathscr{G}_{\mathrm{SL}_n}$. Consider the open cover of $\mathscr{G}_{\mathrm{SL}_n}$ constructed in 4.1. The embedding $G \hookrightarrow \mathrm{SL}_n$ induces the following diagram:

$$\begin{tikzcd} \mathrm{L}^{<0}G \arrow[r, hook, "f"] \arrow[d, "\cong"] & \mathrm{L}^{<0}\mathrm{SL} \arrow[d, "\cong"] \\ \mathcal{U}_{0,G} \arrow[r, "f'"] & \mathcal{U}_0 \end{tikzcd}$$

By construction f is a closed immersion (see Lemma 2.9) and since the vertical morphisms are isomorphisms so is f'.

We observe that for an open subset of LSL of the form $g\cdot \mathrm{L}^{<0}\mathrm{SL}$ with $g \in \mathrm{LSL}$ its preimage in $\mathrm{L}G$ is either empty or we can assume $g \in \mathrm{L}G$ and the preimage is $g \cdot \mathrm{L}^{<0}G$:
If $g \cdot \mathrm{L}^{<0}\mathrm{SL} \cap \mathrm{L}G \neq \emptyset$ then it contains a k-point g'. Then $g' \cdot \mathrm{L}^{<0}\mathrm{SL} = g \cdot \mathrm{L}^{<0}\mathrm{SL}$ and by assumption $g' \in \mathrm{L}G$.

Then multiplication by g induces automorphisms of $\mathrm{L}G$, LSL, $\mathscr{G}_{\mathrm{SL}_n}$ and $\mathscr{G}_G$ and we obtain

$$\begin{tikzcd} g \cdot \mathrm{L}^{<0}G \arrow[r, hook, "f"] \arrow[d, "\cong"] & g \cdot \mathrm{L}^{<0}\mathrm{SL} \arrow[d, "\cong"] \\ g \cdot \mathcal{U}_{0,G} \arrow[r, "f'"] & g \cdot \mathcal{U}_0 \end{tikzcd}$$

With the same argument we get that f' is a closed embedding.

$\square$

4.5
Let $\mathrm{L}^{>2N}G(R)$ be the kernel of the map $G(R[\![z]\!]) \to G(R[z]/(z^{2N}))$. Then $\mathrm{L}^{>2N}G$ is represented by an affine group scheme. We observe that $\mathrm{L}^{>2N}G(R)$ acts trivially on $\mathscr{G}_G^{(N)}(R)$. For $\mathrm{LSL}^{>N}(R)$ this is clear by the lattice-description. Now the assertion is a consequence of Lemma 4.4.

Thus the action of $\mathrm{L}^{\geq 0}G$ on $\mathscr{G}_G^{(N)}$ factors through $\Gamma_N(R) := G\left(R[z]/(z^{2N})\right)$. Γ_N is in an obvious way representable by a k-scheme of finite type.
An analogous statement can be made about $\mathrm{L}^{<0}G \cap \mathrm{L}G^{(N)}$. Its action factors through Γ_N^- defined via $\Gamma_N^-(R) := G\left(R[z^{-1}]/(z^{-2N})\right)$ for the same reason (but the map $\mathrm{L}^{<0}G \cap \mathrm{L}G^{(N)} \to \Gamma_N^-$ is not surjective).

4.6
By Lemma 4.4 we have a closed immersion of the subfunctors $\mathscr{G}_G^{(N)}$:

$$\begin{tikzcd} \mathscr{G}_G \arrow[r, hook] & \mathscr{G}_{\mathrm{SL}_n} \\ \mathscr{G}_G^{(N)} \arrow[u, phantom, "\cup"] \arrow[r, hook] & \mathscr{G}_{\mathrm{SL}_n}^{(N)} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Thus $\mathscr{G}_G^{(N)}$ is a scheme and $\mathscr{G}_G$ is the union of the $\mathscr{G}_G^{(N)}$. Composing with the map from Lemma 3.6 we acquire a closed immersion

$$\mathscr{G}_G^{(N)} \hookrightarrow \mathrm{Grass}^z(2Nn, Nn)$$

which will be helpful in explicit calculations.

5 The Structure of $\mathscr{G}_G$

From now on we assume G to be reductive and connected.

5.1 Affine Schubert varieties
An element $\lambda \in \Lambda_G^+$ corresponds to a point L_λ in $\mathscr{G}_G$ as in 2.11.
The multiplication $\mathfrak{m}\colon \mathrm{L}G \times \mathrm{L}G \to \mathrm{L}G$ induces a transitive $\mathrm{L}G$-action on $\mathscr{G}_G$. Let $\mathscr{G}_\lambda = \mathrm{L}^{\geq 0}G \cdot L_\lambda$ be the orbit of L_λ under the induced $\mathrm{L}^{\geq 0}G$-action. Let $\overline{\mathscr{G}_\lambda}$ be its closure in $\mathscr{G}_G$. We equip it with the reduced scheme structure. Since $\overline{\mathscr{G}_\lambda}$ is $\mathrm{L}^{\geq 0}G$-invariant it is the union of $\mathrm{L}^{\geq 0}G$-orbits.
Since $L_\lambda \in \mathscr{G}_G^{(N)}$ for $N \gg 0$ and since $\mathscr{G}_G^{(N)}$ is closed in $\mathscr{G}_G$ and stable under the $\mathrm{L}^{\geq 0}G$-action we see that $\overline{\mathscr{G}_\lambda} \subset \mathscr{G}_G^{(N)}$ is a projective scheme of finite type over k. It is called *(affine) Schubert variety for* λ.

5.2 The partial order of co-weights
There is a partial order on Λ_G given by $\lambda \leq \mu$ if and only if $\lambda - \mu$ is a sum of simple co-roots. This restricts to a partial order $\leq$ on the dominant co-weights Λ_G^+.
We recall some facts about this order. More details can be found for example in [Humphreys, 1972].
There are only finitely many $\leq$-minimal elements in Λ_G^+, as the lattice Q_G has finite index in the lattice Λ_G. These are 0, the fundamental co-weights that are not co-roots and possibly positive sums of those.
Furthermore every dominant co-weight $\lambda \in \Lambda_G^+$ lies above a unique smallest dominant co-weight λ_0: Let $\lambda \geq \lambda_0$ and $\lambda \geq \lambda_0'$ with λ_0 and λ_0' $\leq$-minimal. By definition this means $\lambda - \lambda_0$ and $\lambda - \lambda_0'$ are positive sums of simple co-roots. Therefore $\lambda_0 - \lambda_0'$ is in the co-root lattice. But since both are $\leq$-minimal $\lambda_0 - \lambda_0'$ is a co-root and therefore 0.
We claim that if the Dynkin diagram I_G of G is connected then above each smallest co-weight, there is a unique "second smallest" co-weight. Clearly the order above each λ_k is isomorphic to the one above 0 by translation with λ_k.
Recall that if I_G is connected, then there is a unique root that is maximal for $\leq$, called the highest root. It is the unique long root that is also a dominant weight. The co-root to the highest root is short and a dominant co-weight (and unique with these properties by the uniqueness of the highest root), call it $\lambda_{\min}$. Then $\lambda_{\min}$ is the unique $\leq$-minimal co-weight above 0. In the simply laced case this is obvious, since $\lambda_{\min}$ is the only root in Λ_G^+. In the non-simply laced case, there are two roots in Λ_G^+, namely $\lambda_{\min}$ and the highest co-root (meaning the highest root for the dual root system). But the highest co-root is bigger than $\lambda_{\min}$ by definition.

We cite the following well-known facts about the affine Grassmannian:

Lemma 5.3

(a) $\mathscr{G}_G = \dot{\bigcup\limits_{\lambda \in \Lambda_G^+}} \mathscr{G}_\lambda$

(b) $\mathscr{G}_\lambda \subset \overline{\mathscr{G}_\mu}$ if and only if $\lambda \leq \mu$.

(c) $\dim \mathscr{G}_\lambda = \langle 2\rho, \lambda \rangle$ where 2ρ is the sum of all positive roots.

Proof. See [Beauville and Laszlo, 1994] for the case of SL_n. For the general case:

(a) Follows from the Cartan decomposition of $G(k(\!(z)\!))$:

$$G(k(\!(z)\!)) = \dot{\bigcup_{\lambda \in \Lambda_G^+}} G(k[\![z]\!]) \cdot \lambda(z) \cdot G(k[\![z]\!])$$

This can be found in [Cartier, 1979, page 140].

(b) See for example [Rapoport, 2005, Notes added June 2003, 2)].

(c) This is proved in [Ngô and Polo, 2000, Lemme 2.2].

□

5.4 Connected components of $\mathscr{G}_G$
Let $\lambda_0 = 0, \lambda_1, \ldots, \lambda_r$ be the minimal dominant coweights as in 5.2. With

$$Y_i := \bigcup_{\substack{\lambda \in \Lambda_G^+ \\ \lambda \geq \lambda_0}} \mathscr{G}_\lambda$$

for $i = 0, \ldots, r$ we have

$$\mathscr{G}_G = \coprod_i Y_i$$

This is an easy consequence of Lemma 5.3: Y_i is closed by b). By a) each $\mathrm{L}^{\geq 0}G$-orbit lies in exactly one Y_i.
If G is semi-simple then $\mathrm{L}^{\geq 0}G$ is connected [Laszlo and Sorger, 1997] and the Y_k are the connected components of $\mathscr{G}_G$ but we will not need this.

Furthermore all Y_k are isomorphic. The isomorphism of Y_0 to Y_i is given by the action of $\lambda_k(z)$. But of course the decomposition into $\mathrm{L}^{\geq 0}G$ orbits is different on each Y_i since multiplication with $\lambda_k(z)$ does not commute with the action of $\mathrm{L}^{\geq 0}G$. For example the singularity ac_p of $\mathscr{G}_{\mathrm{PGL}}$ discussed in 10.3 and in section 11 does not appear in Y_0 (see 10.4).
If $k = \mathbb{C}$ it is shown in [Beauville et al., 1998] that $Y_0(k) = \mathscr{G}_{\tilde{G}}(k)$, where $\tilde{G}$ is the universal covering of G. In 5.6 we will show this in the case of PGL_n.

5.5 Simply connected algebraic groups
We call $\pi_1(G) := \Lambda_G/Q_G$ the *algebraic fundamental group of* G. It is known (see [Fulton and Harris, 1999]) that if $k = \mathbb{C}$, then the algebraic and the topological fundamental group of G coincide.
G is called *simply connected* if $\pi_1(G) = 1$.
This is equivalent to all dominant coweights being ≥ 0. Consequently 0 is the only minimal dominant co-weight and $\mathscr{G}_G$ is connected.
More generally it is shown in [Pappas and Rapoport, 2006, 5.3] that $\pi_1(G)$ is naturally isomorphic to $\pi_0(\mathscr{G}_G)$.

5.6 Connected components of PGL_n
We apply 5.4 to $G = \mathrm{PGL}_n$. The root-system of PGL_n is A_{n-1}. In A_{n-1} the sum of two fundamental co-weights is always a co-root. Therefore we know the minimal fundamental co-weights as in 5.4 are $\lambda_0 = 0$ and $\lambda_i = \check{\omega}_i$ for $i = 1, \ldots, n-1$. Thus $\mathscr{G}_{\mathrm{PGL}_n}$ has n connected components.
The natural map $\mathrm{SL}_n \hookrightarrow \mathrm{PGL}_n$ induces a morphism of schemes

$$f\colon \mathrm{L}^{\geq 0}\mathrm{SL} \hookrightarrow \mathrm{L}^{\geq 0}\,\mathrm{PGL}_n$$

and a morphism of k-spaces

$$\bar{f}\colon \mathscr{G}_{\mathrm{SL}_n} \to \mathscr{G}_{\mathrm{PGL}_n}.$$

We claim that $\bar{f}$ induces a bijection of $\mathscr{G}_{\mathrm{SL}_n}(k)$ onto $Y_0(k)$.
Let $\lambda \in Q_{\mathrm{SL}_n}$ and $\lambda' = \lambda \circ f \in Q_{\mathrm{PGL}_n}$. Clearly $\bar{f}$ is given on $\mathscr{G}_\lambda$ as

$$\begin{aligned} \mathscr{G}_{\mathrm{SL}_n} \supset \mathscr{G}_\lambda &\to \mathscr{G}_{\lambda'} \subset \mathscr{G}_{\mathrm{PGL}_n} \\ g \cdot \lambda(z) &\mapsto f(g) \cdot \lambda'(z) \end{aligned} \tag{4}$$

As all dominant co-weights ≥ 0 in $\Lambda^+_{\mathrm{PGL}_n}$ are of the form λ' for $\lambda \in Q_{\mathrm{SL}_n}$ it is enough to show that the map in (4) is a bijection on closed points.
Injectivity is immediate: if a scalar matrix with entries in $k(\!(z)\!)$ has determinant 1, its entries are already in k. But then it is contained in $\mathrm{SL}_n(k[\![z]\!])$.
Let $g \cdot \lambda'(z) \in \mathscr{G}_{\lambda'}(k)$ with $g \in \mathrm{L}^{\geq 0}\,\mathrm{PGL}_n(k)$. Let $\bar{g} \in \mathrm{L}^{\geq 0}\,\mathrm{GL}_n(k)$ represent g. By 5.1 we can assume $\bar{g}$ to have entries in $k[z]$.
Then $\det(\bar{g}) \in k[z]^\times = k^\times$. Since k is algebraically closed there is $x \in k$ such that $\det(x \cdot \bar{g}) = 1$.
Then $x \cdot \bar{g} \in \mathrm{L}^{\geq 0}\mathrm{SL}(R')$ and obviously $f(x \cdot \bar{g} \cdot \lambda(z)) = g \cdot \lambda'(z)$.

Taking a closer look on the fundamental co-weights of PGL_n there is a more descriptive interpretation of the connected components.
We observe that every element $L \in \mathscr{G}_{\mathrm{PGL}_n}(k)$ can be represented by an element $g \in \mathrm{GL}_n(k(\!(z)\!))$ with determinant z^i such that $0 \leq i < n$. Indeed, let $g \in \mathrm{GL}_n(k(\!(z)\!))$ represent L. Then $\det(g) \in k(\!(z)\!)^\times$ is of the form $u \cdot z^i +$ lower z-terms with $u \in k^\times$. Multiplying with an element of $\mathrm{GL}_n(k[\![z]\!])$ we can choose g such that $\det(g) = z^i$. Since $z \cdot 1_n$ is a scalar matrix with coefficients in $k(\!(z)\!)$

we can assume $0 \leq i < n$.
Every $L \in Y_0 \cong \mathscr{G}_{\mathrm{SL}_n}(k)$ can be represented by a matrix with determinant 1 and we can find representatives of $\omega_i(z)$ with determinant z^i (modulo a permutation). As we know that the isomorphism of Y_0 with Y_i is given by multiplication with $\omega_i(z)$ we conclude that the connected component Y_i is characterized as the subset of points of $\mathscr{G}_{\mathrm{PGL}_n}$ that can be represented by matrices with determinant z^i.

5.7 Minimal degenerations of co-weights
A *minimal degeneration* of co-weights is a pair (λ, μ) with $\lambda, \mu \in \Lambda_G^+$ such that

- $\lambda < \mu$
- if $\lambda \leq \nu \leq \mu$ with $\nu \in \Lambda_G^+$, then $\nu = \lambda$ or $\nu = \mu$.

In other words, λ and μ are neighbours in the partial order on Λ_G^+.

5.8 The singularity belonging to a minimal degeneration
The object of our study will be

$$\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu} \tag{5}$$

where (λ, μ) is a minimal degeneration of co-weights.
In 5.2 arises a first example, the pair $(0, \lambda_{\min})$. Then (5) is the intersection of the second smallest orbit of Y_0 (the smallest non-trivial one) with the open cell (defined in 3.15). This will be discussed in detail in section 8.

Lemma 5.9

$$\mathscr{G}_G = \dot{\bigcup_{\substack{\lambda \in \Lambda_G^+ \\ p \in G \cdot L_\lambda}}} \mathrm{L}^{<0}G \cdot p$$

Proof. The statement is equivalent to

$$\mathscr{G}_G(k) = \dot{\bigcup_{\lambda \in \Lambda_G^+}} G(k[z^{-1}]) \cdot L_\lambda \tag{6}$$

Indeed, $\mathrm{L}^{<0}G(k)$ is defined as the kernel of $G(k[z^{-1}]) \to G(k)$ and thus every element in $G(k[z^{-1}])$ can be expressed uniquely as a product of an element of $\mathrm{L}^{<0}G(k)$ with an element of $G(k)$.
The equation (6) is a consequence of the decomposition

$$G(k(\!(z)\!)) = \dot{\bigcup_{\lambda \in \Lambda_G^+}} G(k[z^{-1}]) \cdot \lambda(z) \cdot G(k[\![z]\!]) \tag{7}$$

The equality (7) is proved in [Faltings, 2003, Lemma 4].
We try to explain the approach used in the proof. The idea is to generalize the follwing characterization of vectorbundles on $\mathbb{P}^1_k$.
Let $\mathbb{A}_0 = \operatorname{Spec} k[z] = D_+(z^{-1}) \subset \mathbb{P}^1_k$ and $\mathbb{A}_\infty = \operatorname{Spec} k[z^{-1}] = D_+(z) \subset \mathbb{P}^1_k$.
A vectorbundle on $\mathbb{P}^1_k$ is given by gluing (trivial) vectorbundles on $\mathbb{A}_0$ and $\mathbb{A}_\infty$ along $\operatorname{Spec} k[z, z^{-1}]$. The gluing isomorphism is given by an automorphism of the trivial bundle of rank n on $\operatorname{Spec} k[z, z^{-1}]$, hence as an element g of $\mathrm{GL}_n(k[z, z^{-1}])$. A change of basis in a free modules of rank n over $k[z]$ and $k[z^{-1}]$ corresponds to the transformation $g \mapsto g_-^{-1} \cdot g \cdot g_+$ with $g_- \in \mathrm{GL}_n(k[z^{-1}])$ and $g_+ \in \mathrm{GL}_n(k[z])$.
Let

$$E := \{\text{Vectorbundles on } \mathbb{P}^1_k\} / \cong$$

It is known that the construction above provides a bijection

$$\psi \colon E \leftrightarrow \mathrm{GL}_n(k[z^{-1}]) \backslash \mathrm{GL}_n(k[z, z^{-1}]) / \mathrm{GL}_n(k[z]).$$

Define

$$\mathbb{Z}^n_+ := \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \cdots \geq m_n\}$$

It is known that any vectorbundle $\mathscr{E}$ of rank n on $\mathbb{P}^1_k$ has a unique decomposition into a direct sum

$$\mathscr{E} \cong \bigoplus_{i=1}^n \mathscr{O}(m_i)$$

with $m_i \in \mathbb{Z}$. This can be found in [Grothendieck, 1957].
Ordering the m_i we obtain a bijection

$$\varphi \colon E \leftrightarrow \mathbb{Z}^n_+$$

Define a map

$$\begin{aligned} p \colon \mathbb{Z}^n_+ &\to \mathrm{GL}_n(k[z^{-1}]) \backslash \mathrm{GL}_n(k[z, z^{-1}]) / \mathrm{GL}_n(k[z]) \\ (m_1, \ldots, m_n) &\mapsto \operatorname{diag}(z^{m_1}, \ldots, z^{m_n}) \end{aligned}$$

We claim $p = \psi \circ \varphi^{-1}$.
Let $(m_1, \ldots, m_n) \in \mathbb{Z}^n_+$ and let $\mathscr{E}$ be the vectorbundle obtained by gluing $\mathscr{O}^n_{\mathbb{A}_0}$ and $\mathscr{O}^n_{\mathbb{A}_\infty}$ along $\operatorname{Spec} k[z, z^{-1}]$ via the isomorphism given by $\operatorname{diag}(z^{m_1}, \ldots, z^{m_n})$ (with respect to the standard basis).
This isomorphism obviously respects the direct sums. Hence $\mathscr{E}$ is the direct sum of the linebundles obtained by gluing $\mathscr{O}_{\mathbb{A}_0}$ and $\mathscr{O}_{\mathbb{A}_\infty}$ along $\operatorname{Spec} k[z, z^{-1}]$ via multiplication with z^{m_i}. It is easy to see that these linebundles are $\mathscr{O}(m_i)$. This proves the claim.
Consequently p is a bijection, in particular surjective. We obtain:

$$\begin{aligned} &G(k[z^{-1}]) \backslash \mathrm{GL}_n(k[z, z^{-1}]) / G(k[z]) \\ &= \dot{\bigcup_{(m_1, \ldots, m_n) \in \mathbb{Z}^n_+}} \mathrm{GL}_n(k[z^{-1}]) \backslash \mathrm{GL}_n(k[z^{-1}]) \cdot \operatorname{diag}(z^{m_1}, \ldots, z^{m_n}) \cdot \mathrm{GL}_n(k[z]) / \mathrm{GL}_n(k[z]) \end{aligned}$$

and

$$\mathrm{GL}_n(k[z, z^{-1}]) = \dot{\bigcup_{(m_1, \ldots, m_n) \in \mathbb{Z}^n_+}} \mathrm{GL}_n(k[z^{-1}]) \cdot \operatorname{diag}(z^{m_1}, \ldots, z^{m_n}) \cdot \mathrm{GL}_n(k[z]) \quad (8)$$

The proof in [Faltings, 2003] modifies this idea in two aspects:

- The gluing of trivial vectorbundles on copies of $\mathbb{A}^1_k$ is replaced by gluing vectorbundles on $\mathbb{P}^1_k \setminus \{\infty\}$ and the formal completion along $\{0\}$.
- Vectorbundles are replaces by G-torsors, to obtain results for reductive groups instead of GL_n

We explain the first modification in the case of vectorbundles:
Observe that $k[z] \to k[z, z^{-1}]$ and $k[z] \to k[\![z]\!]$ are flat, since they are torsion free over $k[z]$. Since

$$\operatorname{Spec} k[z, z^{-1}] \amalg \operatorname{Spec} k[\![z]\!] \to \operatorname{Spec} k[z]$$

is obviously surjective, it is a faithfully flat cover.
This extends to a faithfully flat cover

$$\mathbb{P}^1_k \setminus \{0\} \amalg \operatorname{Spec} k[\![z]\!] \to \mathbb{P}^1_k$$

Using faithfully flat descent (SGA1 [Grothendieck and Raynaud, 1971, Exposé VIII]) we can glue vectorbundles $\mathscr{E}'$ on $\mathbb{P}^1_k \setminus \{0\} \cong \operatorname{Spec} k[z^{-1}]$ and $\mathscr{E}''$ on $\operatorname{Spec} k[\![z]\!]$ to a vectorbundle on $\mathbb{P}^1_k$ by giving an isomorphism of their "restrictions" $\mathscr{E}' \otimes_{k[z]} k(\!(z)\!)$ resp. $\mathscr{E}'' \otimes_{k[z]} k(\!(z)\!)$ to $\operatorname{Spec} k(\!(z)\!)$.
Reformulating this in terms of matrices, this amounts to the following:
A vectorbundle on $\mathbb{P}^1_k$ is given by $g \in \mathrm{GL}_n(k(\!(z)\!))$, corresponding to the isomorphism of $\mathscr{E}_0 \otimes_{k[z]} k(\!(z)\!)$ with $\mathscr{E}'' \otimes_{k[z]} k(\!(z)\!)$. A choice of basis of $\mathscr{E}'$ and $\mathscr{E}''$ corresponds to the transformation $g \mapsto g_-^{-1} \cdot g \cdot g_+$ with $g_- \in \mathrm{GL}_n(k[z^{-1}])$ and $g_+ \in \mathrm{GL}_n(k[\![z]\!])$. One can show that this gives a bijection

$$\psi' \colon E \leftrightarrow \mathrm{GL}_n(k[z^{-1}) \backslash \mathrm{GL}_n(k(\!(z)\!)) / \mathrm{GL}_n(k[\![z]\!])$$

There is a natural map

$$p' \colon \mathrm{GL}_n(k[z^{-1}]) \backslash \mathrm{GL}_n(k[z, z^{-1}]) / \mathrm{GL}_n(k[z]) \to \mathrm{GL}_n(k[z^{-1}]) \backslash \mathrm{GL}_n(k(\!(z)\!)) / \mathrm{GL}_n(k[\![z]\!])$$

Obviously we have $p' \circ \psi = \psi'$. But this means p' is surjective (even bijective) and using (8) we obtain

$$\mathscr{G}(k(\!(z)\!)) = \dot{\bigcup_{(m_1,\ldots,m_n) \in \mathbb{Z}^n_+}} \mathrm{GL}_n(k[z^{-1}]) \cdot \operatorname{diag}(z^{m_1}, \ldots, z^{m_n}) \cdot \mathrm{GL}_n(k(\!(z)\!))$$

$\square$

5.10
It is interesting to note that one can use the construction in the proof of Lemma 5.9 the other way around. It is possible to verify (8) by explicit calculations to obtain another proof of the characterization of vectorbundles on $\mathbb{P}^1_k$. This calculation was first done by Birkhoff. An exposition can be found in [Bodnarchuk et al., 2006].
We could in fact have used the very same calculation to proof our result (at least in the case of GL_n).

5.11 Loop rotation
The so-called "loop rotation" give a useful description of $\mathrm{L}^{<0}G \cdot L_\lambda$.
To define it we first observe that $\mathscr{G}_G$ fulfills the valuative criterion for properness (since $\mathscr{G}_G$ is not a scheme of finite type that does not mean that $\mathscr{G}_G$ is proper in any sense). Let R be a discrete valuation ring with field of fractions K. Given a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "f"] \arrow[d, hook] & \mathscr{G}_G \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

there is a unique morphism $\operatorname{Spec} R \to \mathscr{G}_G$ such that the diagram commutes: The image of f is contained in $\mathscr{G}_G^{(N)}$ for suitable $N \in \mathbb{N}$ by Lemma 2.4. But $\mathscr{G}_G^{(N)}$ is a closed subscheme of a Grassmannian, thus projective. Now we can use the valuative criterion for properness on $\mathscr{G}_G^{(N)} \to \operatorname{Spec} k$ to obtain the desired morphism.

Given $L \in \mathscr{G}_G(k)$ we can view it as an element of $\mathscr{G}_G(k(\!(s)\!))$ via the natural inclusion. Let s^{-1} be the automorphism of $\mathscr{G}_G(k(\!(s)\!))$ induced by the map

$$\begin{aligned} k(\!(s)\!)[\![z]\!] &\to k(\!(s)\!)[\![z]\!] \\ z &\mapsto s^{-1}z \end{aligned}$$

Define $s^{-1}L$ to be the image of L under s^{-1}.
By the valuation criterion this $k(\!(s)\!)$-valued point extends to a $k[\![s]\!]$-valued point (which we also call $s^{-1}L$). Define $\lim\limits_{s\to 0} s^{-1}L \in \mathscr{G}_G(k)$ to be its image under the reduction map $k[\![s]\!] \to k, s \mapsto 0$.
Thus we have defined a map

$$\lim_{s\to 0} s^{-1} \colon \mathscr{G}_G(k) \to \mathscr{G}_G(k(\!(s)\!)) \xrightarrow{s^{-1}} \mathscr{G}_G(k(\!(s)\!)) \to \mathscr{G}_G(k[\![s]\!]) \xrightarrow{z\mapsto 0} \mathscr{G}_G(k)$$

5.12
We have $\lim_{s\to 0} s^{-1} \cdot L_\lambda = L_\lambda$ since

$$\lambda(s^{-1}z) = \lambda(z) \cdot \lambda(s^{-1}) \equiv \lambda(z) \mod G(k(\!(s)\!)[\![z]\!]).$$

Lemma 5.13
Let $L \in \mathscr{G}_G(k)$, let $p \in G(k) \cdot L_\lambda$. Then:

(*a*)
$$L \in \mathrm{L}^{<0}G \cdot p \Leftrightarrow \lim_{s\to 0} s^{-1}L = p$$

(*b*) $\mathrm{L}^{<0}G \cdot L_\lambda$ is s^{-1}-stable in the sense that for $L \in \mathrm{L}^{<0}G \cdot L_\lambda$ we have $s^{-1}L \in \mathrm{L}^{<0}G\,(k(\!(s)\!)) \cdot L_\lambda$.

Proof.

(a) Let $L = h \cdot p \cdot L_\lambda$ with $h \in \mathrm{L}^{<0}G$.
Define $s^{-1}\colon \mathrm{L}^{<0}G(k) \to \mathrm{L}^{<0}G(k(\!(s)\!))$ in the obvious way. Then $s^{-1}h$ is a $k[\![s]\!]$-valued point of $\mathrm{L}^{<0}G$ which allows us to set $s = 0$. Obviously $\lim_{s\to 0} s^{-1}h = 1_G$. Thus

$$\begin{aligned}\lim_{s\to 0} s^{-1}L &= \lim_{s\to 0} s^{-1}(h \cdot p \cdot L_\lambda) \\ &= (\lim_{s\to 0} s^{-1}h) \cdot p \cdot (\lim_{s\to 0} s^{-1}L_\lambda) && \text{since } p \text{ is a } k\text{-point} \\ &= p \cdot L_\lambda && \text{by 5.12}\end{aligned}$$

The other implication follows from 5.9.

(b) This follows from 5.12 and $s^{-1}h \in \mathrm{L}^{<0}G\,(k(\!(s)\!))$ for $h \in \mathrm{L}^{<0}G(k)$

□

Lemma 5.14
For all $\lambda \in \Lambda_G^+$ we have

$$\mathrm{L}^{<0}G \cdot L_\lambda \cap \mathscr{G}_\lambda = \{L_\lambda\}$$

Proof. Let $L \in \mathrm{L}^{<0}G \cdot L_\lambda \cap \mathscr{G}_\lambda$.
Recall the construction of 3.13. Let $\lambda(z) = \mathrm{diag}(z^{\lambda_1}, \ldots, z^{\lambda_n})$. Let $N \in \mathbb{N}$ such that $\mathscr{G}_\lambda \subset \mathscr{G}_G^{(N)}$. Then $z^j e_i$ with $N > j \geq \lambda_i$, $i = 1, \ldots, n$ is a basis of L_λ as a point of $\mathrm{Grass}^z(nN, 2nN)$. Let $A \in \mathrm{M}_{(n,r)}(R)$ be the matrix representing L_λ that corresponds to this basis.
Let $L = g \cdot L_\lambda$ with $g \in G(R[\![z]\!])$. By 4.5 we can assume $g = g_0 + zg_1 + z^2 g_2 + \ldots + z^N g_N$ with $g_i \in \mathrm{M}_n(R)$. Written as $n \times n$ block matrices, representing point of $\mathrm{Grass}^z(nN, 2nN)$ we have

$$L \equiv g \cdot L_\lambda \equiv \begin{pmatrix} g_0 & 0 & \cdots & \cdots & 0 \\ g_1 & g_0 & \ddots & & \vdots \\ g_2 & g_1 & g_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ g_N & g_{N-1} & \cdots & g_1 & g_0 \end{pmatrix} \cdot L_\lambda$$

By the form of the basis given, we know that in every column there is exactly one entry "1". Therefore the matrix $g \cdot L_\lambda \cdot g_0^{-1} \in U_\lambda$ is of the standard form for U_λ (meaning the rows corresponding to the basis given above form a $nN \times nN$ identity matrix, see 3.13).
Written as a $n \times n$ block matrix A consists of blocks of diagonal matrices with

diagonal entries 0 or 1. Therefore

$$L \equiv g \cdot L_\lambda \cdot g_0^{-1} \equiv \begin{pmatrix} 1_n & 0 & \cdots & \cdots & 0 \\ g_1' & 1_n & \ddots & & \vdots \\ g_2' & g_1' & 1_n & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ g_N' & g_{N-1}' & \cdots & g_1' & 1_n \end{pmatrix} \cdot L_\lambda$$

for suitable $g_j' \in \mathrm{M}_n(R)$. Write g' for this block matrix.
We assumed $L \in \mathrm{L}^{<0}G \cdot L_\lambda$. By Lemma 5.13 we know

$$\lim_{s \to 0} s^{-1} \cdot \left(g' \cdot L_\lambda\right) = L_\lambda$$

But $g' \cdot L_\lambda$ differs from L_λ only in entries below the non-zero entries of L_λ. But then the equality above only holds if $g_j' = 0$ for all j. Consequently $L = L_\lambda$. $\square$

5.15
The proof of Lemma 5.14 also yields that $\mathrm{L}^{<0}G \cdot L_\lambda \cap \mathscr{G}_{\lambda'} = \emptyset$ for $\lambda' \ngeq \lambda$. We will not need this, but it justifies the construction of the transverse slice (see Lemma 6.8).

6 Basic properties of $\mathscr{G}_G$ and $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$

6.1 Formal smoothness of k-spaces
Following [Drinfeld, 2003, 6.3.6] we call a morphism of k-spaces $f\colon X \to Y$ *formally smooth* if it satisfies the infinitesimal lifting criterion for affine test-schemes. This means we require for every affine Y-scheme $\operatorname{Spec} A$ and every nilpotent ideal $I \subset A$ the natural map

$$\mathrm{Hom}_Y(\operatorname{Spec} A, X) \to \mathrm{Hom}_Y(\operatorname{Spec} A/I, X)$$

to be surjective.
In other words, in every diagram of the form

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

there is a morphism $g\colon \operatorname{Spec} A \to Y$ such that the diagram is commutative.
Formal smoothness is stable under basechange by the universal property of fibre products.

A morphism of k-spaces $f\colon X \to Y$ is formally smooth if and only if every basechange to an affine k-scheme is formally smooth:
One direction follows since formal smoothness is stable under basechange.
Let $f\colon X \to Y$ such that every basechange $f_B\colon X \times_Y \operatorname{Spec} A \to \operatorname{Spec} B$ to an affine k-scheme $\operatorname{Spec} B$ is formally smooth and let

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

as above. Then in particular the basechange f_A is formally smooth. By the universal property of $X \times_Y \operatorname{Spec} A$ the morphism $\operatorname{Spec}(A/I) \to X$ factors as

$$\begin{tikzcd} \operatorname{Spec}(A/I) \arrow[r] \arrow[d] & X \times_Y \operatorname{Spec} A \arrow[r] \arrow[d, "f_A"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "="] & \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

Since f_A is formally smooth, there is a lifting $g_A\colon \operatorname{Spec} A \to X \times_Y \operatorname{Spec} A$ and hence a lifting $g\colon \operatorname{Spec} A \to X$.

If $f\colon X \to Y$ is representable then smoothness is local on the base. This means if there is an open cover U_i of Y such that $f_i\colon X \times_Y U_i \to U_i$ is formally smooth, then f is formally smooth.
We have seen that it es enough to check smoothness for every basechange f_B to an affine k-scheme $\operatorname{Spec} B$. If U_i is an open cover of Y, then $U_i \times_Y \operatorname{Spec} B$ is an open cover of $\operatorname{Spec} B$. If f_i is formally smooth, so is

$$f_{i,B}\colon X \times_Y U_i \times_Y \operatorname{Spec} B \to U_i \times_Y \operatorname{Spec} B$$

If we assume f to be representable, then $X \times_Y \operatorname{Spec} B$ is a scheme, f_B is a morphism of schemes and $f_{i,B}$ are the restrictions to the open cover $U_i \times_Y \operatorname{Spec} B$ of $\operatorname{Spec} B$. Since formal smoothness is local for morphisms of schemes (see EGA IV_4 [Grothendieck and Dieudonné, 1967, Proposition 17.1.6]), we find that f_B is formally smooth, proving the statement.

Lemma 6.2
$\mathrm{L}^{\geq 0}G$ is formally smooth over k.

Proof. We have to prove that for every k-algebra A and every nilpotent ideal I in A the natural map

$$\mathrm{L}^{\geq 0}G(A) = G(A[\![z]\!]) \to G((A/I)[\![z]\!]) = \mathrm{L}^{\geq 0}G(A/I)$$

is surjective.
Let $\tilde{I}$ be the kernel of the natural surjection $A[\![z]\!] \twoheadrightarrow (A/I)[\![z]\!]$.
If $I^r = (0)$ then $\tilde{I}^r = (0)$. Thus $\tilde{I}$ is a nilpotent ideal of $A[\![z]\!]$ and

$$G(A[\![z]\!]) \to G(A[\![z]\!]/\tilde{I}) = G((A/I)[\![z]\!])$$

is surjective since G is smooth over k. $\square$

Lemma 6.3
The quotient map $q\colon \mathrm{L}G \to \mathrm{L}G/\mathrm{L}^{\geq 0}G = \mathscr{G}_G$ is formally smooth.

Proof. First we prove that q is representable. By 4.3 the basechange of q to the open cell $\mathrm{L}^{<0}G \cdot L_0$ is the first projection

$$\mathrm{pr}_1\colon \mathrm{L}^{<0}G \times \mathrm{L}^{\geq 0}G \to \mathrm{L}^{<0}G \cdot L_0$$

Let $S \to \mathscr{G}_G$ with S a scheme. We have to show that the fibre product $S \times_{\mathscr{G}_G} \mathrm{L}G$ is representable.
Let $S' = S \times_{\mathscr{G}_G} \mathrm{L}^{<0}G \cdot L_0$. Then S' is an open subscheme of S and there is a cartesian diagram

$$\begin{tikzcd}
\mathrm{L}^{\geq 0}G \times_k S' \arrow[r] \arrow[d] & S' \arrow[d] \\
\mathrm{L}^{<0}G \times_k \mathrm{L}^{\geq 0}G \arrow[r] & \mathrm{L}^{<0}G \cdot L_0
\end{tikzcd}$$

Since $\mathrm{L}^{\geq 0}G$ is a scheme, so is $\mathrm{L}^{\geq 0}G \times_k S'$. Doing the same for the translates of $\mathrm{L}^{<0}G$ we obtain an open cover of $S \times_{\mathscr{G}_G} \mathrm{L}G$ by schemes of the form $\mathrm{L}^{\geq 0}G \times_k S'$. Gluing them together yields that $S \times_{\mathscr{G}_G} \mathrm{L}G$ is representable.
Using 6.1 we can test whether q is formally smooth on the open cover formed by the translates of the open cell $\mathrm{L}^{<0}G \cdot L_0$. As above the basechange to $\mathrm{L}^{<0}G \cdot L_0$ is just a projection which is a basechange of the structure morphism $\mathrm{L}^{\geq 0}G \to \operatorname{Spec} k$. But this is smooth by Lemma 6.2. $\square$

For the next proof we need the following notions. They are discussed in detail for example in [Altman and Kleiman, 1970].

6.4 Serre's conditions R_l and S_l
A locally noetherian scheme X is said to *satisfy condition* R_l for $l \in \mathbb{N}$ if X is regular in co-dimension $\leq l$, i.e. if all local rings $\dim \mathscr{O}_{X,x}$ with $\dim \mathscr{O}_{X,x} \leq l$ are regular.
X is said to *satisfy condition* S_l for $l \in \mathbb{N}$ if for all $x \in X$

$$\mathrm{depth}(\mathscr{O}_{X,x}) \geq \inf\{l, \dim(\mathscr{O}_{X,x})\}.$$

For example X satisfies R_l for all $l \in \mathbb{N}$ if and only it is regular. It satisfies S_l for all $l \in \mathbb{N}$ if and only if it is Cohen-Macaulay. It is known that X is normal if and only if it satisfies R_1 and S_2. This is called "Serre's criterion" and can also be found in [Altman and Kleiman, 1970].

Lemma 6.5 ([Altman and Kleiman, 1970, Theorem 4.8])
Let X and Y be locally noetherian schemes and $f\colon X \to Y$ a faithfully flat morphism. Fix $l \in \mathbb{N}$. Then:

(a) If X satisfies R_l (resp. S_l), then Y satisfies R_l (resp. S_l).

(b) If Y satisfies R_l (resp. S_l) and for each $y \in f(X)$ the fibre over y satisfies R_l (resp. S_l), then X satisfies R_l (resp. S_l).

6.6
We will use the fact that $\overline{\mathscr{G}_\lambda}$ is normal to prove normality of the scheme $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\lambda}$. The former is proved in [Faltings, 2003, section 4] or in a more general case in [Pappas and Rapoport, 2006, theorem 8.4] , both using Frobenius splitting. This result is not needed in what follows.

Lemma 6.7 ([Malkin et al., 2005, Lemma 2.5])

(a) Let $\lambda \in \Lambda_G^+$. Then $\mathrm{L}^{<0}G \cdot L_\lambda$ is a transverse slice to $\mathscr{G}_\lambda$ at the point L_λ in the sense that

- (i) $\mathrm{L}^{<0}G \cdot L_\lambda$ is locally closed in $\mathscr{G}_G$
- (ii) the action map $\mathfrak{m} : \mathrm{L}^{\geq 0}G \times \mathrm{L}^{<0}G \cdot L_\lambda \to \mathscr{G}_G$ is formally smooth
- (iii) for $\lambda, \mu \in \Lambda_G^+$ with $\lambda \leq \mu$ one has

$$\dim(\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) = \dim \mathscr{G}_\mu - \dim \mathscr{G}_\lambda$$

(b) The scheme $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ is integral and normal.

Proof. (a) Consider the following diagram:

$$\begin{tikzcd}
\{e\} \times \mathrm{L}^{<0}G \arrow[r, hook] \arrow[d, "\cdot L_\lambda"] & \mathrm{L}^{\geq 0}G \times \mathrm{L}^{<0}G \arrow[r, hook, "\mathfrak{m}"] & \mathrm{L}G \arrow[r, "\cong", "\cdot\lambda(z)"'] & \mathrm{L}G \arrow[d] \\
\mathrm{L}^{<0}G \cdot L_\lambda \arrow[rrr] & & & \mathrm{L}G/\mathrm{L}^{\geq 0}G = \mathscr{G}_G
\end{tikzcd}$$

The top row is a closed immersion followed by an open immersion (Lemma 4.2) and since the right vertical map is a topological quotient map (restricted to $\mathscr{G}_G^{(n)}$, commutativity yields i).
Now to ii) It suffices to show that $\mathfrak{m}\colon \mathfrak{m}^{-1}(\mathscr{G}_G^{(N)}) \to \mathscr{G}_G^{(N)}$ is smooth for every $N \in \mathbb{N}$ since every morphism of a quasi-compact scheme to $\mathscr{G}_G$ factors through one of the $\mathscr{G}_G^{(N)}$ (compare Lemma 2.4) and we are only using affine test schemes.
So we have to show that in every diagram of the form

$$\begin{tikzcd}
\left(\Gamma_N \times \mathrm{L}^{<0}G \cdot L_\lambda\right) \cap \mathfrak{m}^{-1}\left(\mathscr{G}_G^{(N)}\right) \arrow[r, "\mathfrak{m}"] & \mathscr{G}_G^{(N)} \\
\mathrm{Spec}(A/I) \arrow[u, "f"] \arrow[r] & \mathrm{Spec}\, A \arrow[u, "g"'] \arrow[ul, dashed, "g'"]
\end{tikzcd}$$

where I is a nilpotent ideal there exists a lift g' of g. By Lemma 4.5 we can replace $\mathrm{L}^{\geq 0}G$ by Γ_N, since the map $\mathrm{L}^{\geq 0}G \to \Gamma_N$ is formally smooth, using Lemma 6.2. Then we can assume $\mathfrak{m}$ to be of finite type and replace formal smoothness with smoothness.

By SGA1 ([Grothendieck and Raynaud, 1971, Exposé III, Corollaire 2.2]) we can then take a local artin ring with residue field k as test scheme. We simplify the notation by omitting the restriction to $\mathscr{G}_G^{(N)}$ and just assume that $\mathfrak{m}$ is of finite type. So it remains to show that in the diagram

$$\begin{tikzcd}
\mathrm{L}^{\geq 0}G \times \mathrm{L}^{<0}G \arrow[r, hook, "\mathfrak{m}"] \arrow[d, two heads, "q"] & \mathrm{L}G \arrow[r, "\cong", "\cdot\lambda(z)"'] & \mathrm{L}G \arrow[d, two heads] \\
\Gamma_N \times \mathrm{L}^{<0}G \cdot L_\lambda \arrow[rr] & & \mathscr{G}_G \\
\operatorname{Spec} k \arrow[u, "f"] \arrow[rr] & & \operatorname{Spec} C \arrow[u, "g"'] \arrow[ull, dashed, "g'"]
\end{tikzcd}$$

there exists a lift g' of g.
Since k is algebraically closed and the fibres of q are isomorphic to $\mathrm{Stab}_{\mathrm{L}^{\geq 0}G}(L_{\lambda(z)})$, the stabilizer of L_λ in $\mathrm{L}^{\geq 0}G$, we can lift f to a geometric point of $\mathrm{L}^{\geq 0}G \times \mathrm{L}^{<0}G$. But since the top row and the quotient map $\mathrm{L}G \to \mathrm{L}G/\,\mathrm{L}^{\geq 0}G$ are formally smooth by Lemma 6.3 we can lift g to a map

$$g'' : \operatorname{Spec} C \to \mathrm{L}^{\geq 0}G \times \mathrm{L}^{<0}G$$

and thus to a map

$$g' : \operatorname{Spec} C \to \mathrm{L}^{\geq 0}G \times \mathrm{L}^{<0}G \cdot L_\lambda.$$

To prove *iii*) we factor the map

$$\mathfrak{m}\colon\ \mathrm{L}^{\geq 0}G \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) \to \overline{\mathscr{G}_\mu}$$

as above, using that $\overline{\mathscr{G}_\mu} \subset \mathscr{G}_G^{(N)}$ for suitable $N \in \mathbb{N}$ (see 5.1):

$$\mathfrak{m}' : \Gamma_N \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) \to \overline{\mathscr{G}_\mu}$$

We claim that the fibre of $\mathfrak{m}'$ over the point $L_\lambda \in \overline{\mathscr{G}_\mu}$ is $\mathrm{Stab}_{\mathrm{L}^{\geq 0}G}(L_\lambda)/\,\mathrm{L}^{>2N}G$. Then

$$\begin{aligned}
&\dim \Gamma_N + \dim(\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) \\
&= \dim \overline{\mathscr{G}_\mu} + \dim(\mathrm{Stab}_{\mathrm{L}^{\geq 0}G}(L_\lambda)/\,\mathrm{L}^{>2N}G)
\end{aligned}$$

since $\mathfrak{m}'$ is smooth by *ii*) and $\mathfrak{m}'$ is of finite type (as in *ii*)). Thus

$$\begin{aligned}
&\dim(\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) \\
&= \dim \overline{\mathscr{G}_\mu} + \dim(\mathrm{Stab}_{\mathrm{L}^{\geq 0}G}(L_\lambda)/\,\mathrm{L}^{>2N}G) - \dim \mathrm{L}^{\geq 0}G/\,\mathrm{L}^{>2N}G \\
&= \dim \overline{\mathscr{G}_\mu} + \dim \overline{\mathscr{G}_\lambda}
\end{aligned}$$

where the last equality holds because $\mathscr{G}_\lambda \cong \mathrm{L}^{\geq 0}G/\,\mathrm{Stab}_{\mathrm{L}^{\geq 0}G}(L_\lambda)$.
To prove the claim we observe that any preimage of L_λ under $\mathfrak{m}$ is of course contained in $\mathscr{G}_\lambda$. Thus it suffices to show $\mathrm{L}^{<0}G \cdot L_\lambda \cap \mathscr{G}_\lambda = \{L_\lambda\}$, which is Lemma 5.14. This proves *iii*).

(*b*) We use the map

$$\mathfrak{m} : \mathrm{L}^{\geq 0}G \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) \to \overline{\mathscr{G}_\mu}.$$

By *a*) *ii*) it is formally smooth.
Locally $\mathrm{L}^{\geq 0}G \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu})$ is of the form $\mathrm{Spec}(A \otimes_k B)$ where $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ are open affine subsets of $\mathrm{L}^{\geq 0}G$ and $(\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu})$ respectively.
Since $\mathrm{L}^{\geq 0}G \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu})$ is integral $A \otimes_k B$ is an integral k-algebra. This is only possible if B is as well integral. Therefore $L_\lambda \cap \overline{\mathscr{G}_\mu}$ is integral.

For normality we factor $\mathfrak{m}$ as above through $\Gamma_N \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu})$ and get a map of finite type that is smooth. Thus it has regular fibres.
This means the fibres satisfy all the R_l and S_l defined in 6.4. Since $\overline{\mathscr{G}_\mu}$ is normal (6.6) it satisfies R_1 and S_2 by Serre's criterion. Using Lemma 6.5 *b*) (with Y the image of $\mathfrak{m}$, which is normal since it is open in $\overline{\mathscr{G}_\mu}$) we conclude that the same is true for $\Gamma_N \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu})$.

The projection $\Gamma_N \times \mathrm{L}^{<0}G \cdot L_\lambda \to \mathrm{L}^{<0}G \cdot L_\lambda$ is a basechange of the structure morphism of Γ_N and therefore faithfully flat. Using Lemma 6.5 *b*) we get that $\mathrm{L}^{<0}G \cdot L_\lambda$ satisfies S_2 and R_1. Using Serre's criterion again yields that it is normal, proving *iii*).

□

6.8 The transverse slice
Let (λ, μ) be a minimal degeneration of co-weights. Using Lemma 5.14, 5.15 and Lemma 6.7 we can explain the idea of the transverse slice $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ defined in 5.8, as indicated in the introduction.
Let $N \in \mathbb{N}$ such that $\mathscr{G}^{(N)}$ contains $\mathscr{G}_\mu$. Then

$$\mathfrak{m}' : \Gamma_N \times (\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}) \to \overline{\mathscr{G}_\mu}$$

is smooth by the proof of Lemma 6.7 and the image meets and therefore contains $\mathscr{G}_\mu$.
By Lemma 5.14 we know that $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ contains L_λ, therefore the image of $\mathfrak{m}'$ also contains $\mathscr{G}_\lambda$. By 5.15 it contains no other orbit, so the image is $\mathscr{G}_\mu \cup \mathscr{G}_\lambda$.
Since Γ_N is smooth over k (argue as in Lemma 6.2), this means $(\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu})$ is smoothly equivalent to $\mathscr{G}_\mu \cup \mathscr{G}_\lambda$.

7 Reduction to the Levi-subgroup

Let $P \subset G$ be a parabolic subgroup with $B \subset P$. Let G_L be a Levi subgroup of P such that $T \subset G_L$. Let $M = [G_L, G_L]$, the commutator of G_L. M is connected and semi simple. Let $I_M \subset I_G$ be the Dynkin diagram associated to M, $\Lambda_M \subset \Lambda_G$ the co-weight lattice of M, $\Lambda_M^+ \subset \Lambda_G^+$ the set of dominant co-weights and Q_M the co-root lattice of M.

For $\lambda \in \Lambda_G$ with

$$\lambda = \sum_{i \in I_G} \lambda_i \check{\omega}_i$$

with $\lambda_i \in \mathbb{Z}$ we define

$$\lambda_M = \sum_{i \in I_M} \lambda_i \check{\omega}_i.$$

Lemma 7.1
Let λ be a co-weight of G such that λ is in the co-weight lattice Λ_M of M. Then

$$\lambda = \lambda_M \in \Lambda_M$$

Proof. Let $(\check{\omega}_i)_{i \in I}$ be the fundamental co-weights of G and $(\check{\omega}_{i,M})_{i \in I_M}$ with $I_M \subset I$ the fundamental co-weights of M.
For $i \in I_M$ the co-roots α_i and $\alpha_{i,M}$ are equal and so

$$\lambda = \sum_{i \in I} \langle \lambda, \check{\alpha}_i \rangle \check{\omega}_i \text{ and}$$
$$\lambda_M = \sum_{i \in I_M} \langle \lambda, \check{\alpha}_i \rangle \check{\omega}_{i,M}$$

For $j \in I_M$ we obtain

$$\langle \lambda_M, \check{\alpha}_j \rangle = \sum_{i \in I_M} \langle \lambda, \check{\alpha}_i \rangle \langle \check{\omega}_{i,M}, \alpha_j \rangle = \langle \lambda, \check{\alpha}_j \rangle$$

Therefore $\langle \lambda - \lambda_M, \check{\alpha}_j \rangle = 0$ for $j \in I_M$.
Since $\lambda - \lambda_M \in \Lambda_M$ and the $\check{\alpha}_i$ are dual to a basis of Λ_M the claim follows. $\square$

7.2
Let U_{α_i}, $i \in I_M$ be the root subgroup associated to α_i (see [Springer, 1998] for example). For any (algebraically closed) field K the group $M(K)$ is generated by the $U_{\pm\alpha_i}(K)$, $i \in I_M$ and the torus of M. Let $\alpha = \pm\alpha_i$, $i \in I_M$.
By the very definition of U_α we have $\check{\omega}_j(z) \cdot x \cdot \check{\omega}_j(z^{-1}) = \alpha(\check{\omega}_j(z)) \cdot x$ for $x \in U_\alpha(K)$. But the composition $\alpha \circ \check{\omega}_j$ is 1 if $j \notin I_M$. Thus conjugation by $(\lambda_M - \lambda)(z)$ induces the identity on all $U_{\pm\alpha_i}(K)$ and hence on $M(K)$.
Using this fact on the algebraic closure of $k(\!(z)\!)$ we find that this conjugation acts trivially on $M(k(\!(z)\!))$ and thus on $M(k[\![z]\!])$.
It follows that $\lambda(z) \cdot \mathrm{L}^{\geq 0} M \cdot \lambda(z^{-1}) = \lambda_M(z) \cdot \mathrm{L}^{\geq 0} M \cdot \lambda_M(z^{-1})$.

Lemma 7.3 ([Malkin et al., 2005, Lemma 3.2])
Let $\lambda \in \Lambda_G^+$. Then there is a natural isomorphism of k-spaces

$$\mathrm{L}\, M \cdot L_\lambda \xrightarrow{\cong} \mathscr{G}_M$$
$$m \cdot L_\lambda \mapsto m \cdot L_{\lambda_M}.$$

Proof. We reproduce the proof of [Malkin et al., 2005].
Knowing that $\mathrm{L}\,M \cdot L_\lambda \cong \mathrm{L}\,M / \operatorname{Stab}_{\mathrm{L}\,M}(L_\lambda)$, we have

$$\begin{aligned}
\operatorname{Stab}_{\mathrm{L}\,M}(L_\lambda) &= \mathrm{L}\,M \cap \big(\lambda(z) \cdot \mathrm{L}^{\geq 0}G \cdot \lambda(z^{-1})\big) \\
&= \lambda(z) \cdot \Big(\big(\lambda(z^{-1}) \cdot \mathrm{L}\,M \cdot \lambda(z)\big) \cap \mathrm{L}^{\geq 0}G\Big) \cdot \lambda(z^{-1}) \\
&= \lambda(z) \cdot \big(\mathrm{L}\,M \cap \mathrm{L}^{\geq 0}G\big) \cdot \lambda(z^{-1}) \\
&= \lambda(z) \cdot \mathrm{L}^{\geq 0}M \cdot \lambda(z^{-1}) \\
&= \lambda_M(z) \cdot \mathrm{L}^{\geq 0}M \cdot \lambda_M(z^{-1}) && \text{by 7.2}
\end{aligned}$$

□

Let $\mathscr{G}_{M,\mu} := \mathrm{L}^{\geq 0}M \cdot L_\mu \subset \mathscr{G}_M$ for $\mu \in \Lambda_M^+$, the Schubert variety of M corresponding to μ.

7.4
Let $\lambda \leq \mu \in \Lambda_G^+$ with $\mu - \lambda \in Q_M$.
Call the isomorphism in Lemma 7.3 f. Then $f(\mathrm{L}^{<0}M \cdot L_\lambda) = \mathrm{L}^{<0}M \cdot L_{\lambda_M}$ and $f(\mathrm{L}^{\geq 0}M \cdot L_\mu) = \mathrm{L}^{\geq 0}M \cdot L_{\mu_M} = \mathscr{G}_{M,\mu}$ by the definition of f. Since f is an isomorphism $f(\overline{\mathrm{L}^{\geq 0}M \cdot L_\mu}) = \overline{\mathrm{L}^{\geq 0}M \cdot L_{\mu_M}}$.
Therefore f restricts to an isomorphism of schemes

$$\mathscr{G}_G \supset (\mathrm{L}^{<0}M \cdot L_\lambda) \cap \overline{\mathrm{L}^{\geq 0}M \cdot L_\mu} \xrightarrow{\cong} (\mathrm{L}^{<0}M \cdot L_{\lambda_M}) \cap \overline{\mathscr{G}_{M,\mu_M}} \subset \mathscr{G}_M.$$

Lemma 7.5
Let $\lambda \leq \mu \in \Lambda_G^+$ with $\mu - \lambda \in Q_M$. Then

$$\dim \mathscr{G}_{M,\mu_M} - \dim \mathscr{G}_{M,\lambda_M} = \dim \mathscr{G}_\mu - \dim \mathscr{G}_\lambda$$

Proof. By Lemma 5.3 *c*) we know that $\dim \mathscr{G}_\lambda = \langle 2\rho, \lambda\rangle$ and $\dim \mathscr{G}_\mu = \langle 2\rho, \mu\rangle$ and similar for $\mathscr{G}_{M,\lambda_M}$ and $\mathscr{G}_{M,\mu_M}$. Thus

$$\begin{aligned}
\dim \mathscr{G}_\mu - \dim \mathscr{G}_\lambda &= \langle 2\rho, \mu\rangle - \langle 2\rho, \lambda\rangle \\
&= \langle 2\rho, \mu - \lambda\rangle \\
&= \langle 2\rho, \mu_M - \lambda_M\rangle && \text{by Lemma 7.1 for } \lambda - \mu \\
&= \langle 2\rho_M, \mu_M - \lambda_M\rangle && \text{as } \langle \check{\alpha}_i, \mu_M - \lambda_M\rangle = 0 \text{ for } i \in I \setminus I_M \\
&= \dim \mathscr{G}_{M,\mu_M} - \dim \mathscr{G}_{M,\lambda_M}
\end{aligned}$$

□

Lemma 7.6 ([Malkin et al., 2005, Lemma 3.3])
Let $\lambda \leq \mu \in \Lambda_G^+$ with $\mu - \lambda \in Q_M$. Then

$$\mathrm{L}^{<0}M \cdot L_\lambda \cap \overline{\mathrm{L}^{\geq 0}M \cdot L_\mu} = \mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathrm{L}^{\geq 0}G \cdot L_\mu}$$

as subschemes of $\mathscr{G}_G$.

Proof. Let $Y_M = \mathrm{L}^{<0}M \cdot L_\lambda \cap \overline{\mathrm{L}^{\geq 0}M \cdot L_\mu}$ and $Y_G = \mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathrm{L}^{\geq 0}G \cdot L_\mu}$. It follows immediately from Theorem 4.4 that $Y_M \subset Y_G$ is an immersion. Since both are equipped with the reduced scheme structure it is enough to show that $Y_G \setminus Y_M = \emptyset$. By Lemma 7.4 $\big($and Lemma 6.7 *a*), *iii*) $\big)$ we know:

$$\begin{aligned}
\dim Y_M &= \dim \Big((\mathrm{L}^{\leq 0} M \cdot \lambda_M(z)) \cap \overline{\mathrm{L}^{\geq 0} M \cdot \mu_M(z)} \Big) \\
&= \dim \mathscr{G}_{M,\mu_M} - \dim \mathscr{G}_{M,\lambda_M} \quad \text{using Lemma 5.3 } c) \\
&= \dim \mathscr{G}_\mu - \dim \mathscr{G}_\lambda \qquad\qquad \text{by Lemma 7.5} \\
&= \dim Y_G
\end{aligned}$$

Y_G is irreducible by Lemma 6.7 *b*), yielding $\overline{Y_M} = \overline{Y_G}$ and

$$Y_G \setminus Y_M \subset \overline{Y_M} \setminus Y_M = \partial Y_M$$

Hence Y_M is locally closed in $\mathscr{G}_G$ by Lemma 6.7 *a*) *i*) and Theorem 4.4. Thus ∂Y_M is closed in $\mathscr{G}_G$.
Assume there is a $L \in Y_G \setminus Y_M$.
Because of Lemma 5.13 we know that $\lim\limits_{s \to 0} s^{-1} L = L_\lambda$. But ∂Y_M does not contain L_λ and is closed. This contradicts the fact that ∂Y_M is s^{-1}-stable by Lemma 5.13 and specialization does not leave a closed subset. □

8 Intersection with the open cell

In this paragraph we assume that G is semi simple and that the Dynkin-diagram of G is connected. We use the notation from 5.2 and study the intersection defined in 5.8, belonging to the minimal degeneration $(0, \lambda_{\min})$. We show that it is a Kleinian singularity arising in the closure of a conjugacy class in the Lie algebra of G.
We do not require G to be simply connected but the singularity $\mathrm{L}^{<0}G \cdot L_0 \cap \overline{\mathscr{G}_{\lambda_{\min}}}$ depends only on the connected component Y_0 containing L_0, since $\mathrm{L}^{<0}G$ is connected (see [Laszlo and Sorger, 1997]).

8.1 Conjugacy classes in the Lie algebra
We recall some facts about conjugacy classes of $\mathfrak{g} = \mathrm{Lie}\, G$.
The embedding $G \hookrightarrow \mathrm{SL}_n$ induces an embedding $\mathfrak{g} \hookrightarrow \mathfrak{sl}_n = \mathrm{Lie}\,\mathrm{SL}_n$. We identify $\mathfrak{sl}_n$ with the matrices in M_n with trace 0. G acts on $\mathfrak{g}$ via the adjoint representation. Using the embedding into $\mathfrak{sl}_n$ this is given by conjugation of matrices

$$\begin{aligned}
G \times \mathfrak{g} &\to \mathfrak{g} \\
(g, x) &\mapsto g \cdot x \cdot g^{-1}
\end{aligned}$$

The orbits of this action are called *conjugacy classes*. The conjugacy classes are locally closed in $\mathfrak{g}$ and their closure is clearly the union of conjugacy classes.
We are interested in the conjugacy classes contained in the nilpotent cone in $\mathfrak{g}$

(resp. $\mathfrak{sl}_n$), defined by the condition that the characteristic polynomial is T^n. These conjugacy classes are discussed in detail in [Kraft and Procesi, 1981] and [Kraft and Procesi, 1982].
There is always a unique closed conjugacy class C and a unique conjugacy class $C_{\min}$ such that $\overline{C_{\min}} = C \cup C_{\min}$. $C_{\min}$ is called the *minimal conjugacy class*, since it is the "smallest" non-trivial one.

8.2
Lemma 3.14 holds for G in the sense that $\mathrm{L}^{<0}G \cdot L_0 \cap \mathscr{G}_G^{(N)}$ is contained in a standard open set of a Grassmannian:

$$\mathrm{L}^{<0}G \cdot L_0 \cap \mathscr{G}_G^{(N)} = \mathcal{U}_{0,G}^{(N)} \hookrightarrow U_0 \cap \mathrm{Grass}^z(nN, 2nN)$$

This gives a description of $\mathrm{L}^{<0}G \cdot L_0 \cap \mathscr{G}_G^{(N)}$ in terms of matrices. This is just the restriction of the statement of Lemma 3.14 to $\mathscr{G}_G$.
There is another account of $\mathcal{U}_{0,G}^{(N)}$:
Obviously $\mathcal{U}_{0,G}^{(N)} \cong \mathrm{L}^{<0}G \times_{\mathscr{G}_G} \mathscr{G}_G^{(N)} =: X$ since $\mathcal{U}_{0,G} \cong \mathrm{L}^{<0}G$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "op."] \arrow[d, "cl."] & \mathrm{L}G^{(N)} \arrow[r, two heads] \arrow[d, "cl."] & \mathscr{G}_G^{(N)} \arrow[d, "cl."] \\ \mathrm{L}^{<0}G \arrow[r, "op."] & \mathrm{L}G \arrow[r, two heads] & \mathscr{G}_G \end{tikzcd}$$

where the right square and the outer rectangle are cartesian, and consequently the left square, too. Here *cl.* stands for "closed immersion" and *op.* for "open immersion".
We obtain

$$X = \mathrm{L}^{<0}G \times_{\mathrm{L}G} \mathrm{L}G^{(N)} \tag{1}$$

Since $\mathrm{L}^{<0}G \times_{\mathrm{L}G} \mathrm{L}G^{(N)} \to \mathscr{G}_G$ embeds into Γ_N^- by 4.5, the diagram

$$\begin{tikzcd} X \arrow[r, "op."] \arrow[d, "cl."] & \Gamma_N^- \arrow[d, "cl."] \\ \mathrm{L}^{<0}G \arrow[r, "op."] & \mathscr{G}_G \end{tikzcd}$$

is also cartesian. Consequently $X \to \Gamma_N^-$ is an open immersion.
By (1) we know $X(R) = \mathrm{L}^{<0}G(R) \cap \mathrm{L}G^{(N)}(R)$. Then

$$X(R) = \ker\Big(G(R[z^{-1}]/z^{-2N}) \to G(R)\Big)$$

since this is clearly the image of $\mathrm{L}^{<0}G(R) \cap \mathrm{L}G^{(N)}(R)$ in $\Gamma_N^-(R) = G\left(R[z^{-1}]/z^{-2N}\right)$.

Theorem 8.3 ([Malkin et al., 2005, 2.10])
There is a natural isomorphism of k-schemes

$$(\mathrm{L}^{<0}G \cdot L_0) \cap \overline{\mathscr{G}_{\lambda_{\min}}} \xrightarrow{\cong} \overline{C_{\min}} \subset \mathfrak{g}$$

Proof. Using 8.2 for $N = 2$ we find that

$$f\colon \left(\mathrm{L}^{<0}G \cdot L_0 \cap \mathscr{G}_G^{(1)}\right)(R) \xrightarrow{\cong} \ker\Big(G(R[z^{-1}]/z^{-2}) \to G(R)\Big)$$

But the right hand side is equal to $\mathfrak{g}(R)$.
We can give this isomorphism more explicitly:
By 8.2 we can write $L \in \mathrm{L}^{<0}G \cdot L_0 \cap \mathscr{G}_G^{(1)}$ as an element of $U_0 \cap \mathrm{Grass}^z(2n, n)$.
This means L is represented by a block matrix

$$L = \begin{pmatrix} A \\ 1_n \end{pmatrix}$$

with respect to the basis $z^{-1}e_1, \ldots, z^{-1}e_n, e_1, \ldots, e_n$ of $(R[z]/z^2)^n$.
By construction (3.12) $U_0 \cong \mathbb{A}^{n^2} = \mathrm{M}_n$. Then f is given by

$$\begin{aligned} f\colon (\mathrm{L}^{<0}G \cdot L_0) \cap \mathscr{G}_G^{(1)} &\hookrightarrow \mathrm{M}_n \\ \left(\begin{smallmatrix} A \\ 1_n \end{smallmatrix}\right) &\mapsto A \end{aligned}$$

To prove the theorem we take a look at the $G(R[\![z]\!])$-action on both sides.
By Lemma 4.5 we know that $G(R[\![z]\!])$ acts on $\mathscr{G}_G^{(1)}$ via $G(R[z]/(z^2))$.
Let $g \in G(R[\![z]\!]), g = g_0 + zg_1$ with $g_0 \in G(R), g_1 \in \mathrm{M}_n(R)$, such that $g \cdot L \in \mathrm{L}^{<0}G \cdot L_0$. Then

$$\begin{aligned} g \cdot \begin{pmatrix} A \\ 1_n \end{pmatrix} &= (g_0 + zg_1) \cdot \begin{pmatrix} A \\ 1_n \end{pmatrix} = \begin{pmatrix} g_0 A \\ g_0 + g_1 A \end{pmatrix} \\ &= \begin{pmatrix} (g_0 + g_1 A)A(g_0 + g_1 A)^{-1} \\ 1_n \end{pmatrix} \cdot (g_0 + g_1 A) \end{aligned}$$

where the last equality holds because $A^2 = 0$. We know that $(g_0 + g_1 A)$ is invertible by the assumption that $g \cdot L \in \mathrm{L}^{<0}G \cdot L_0$.
This means, given $L \in (\mathrm{L}^{<0}G \cdot L_0) \cap \mathscr{G}_G^{(1)}$ then $f'(g \cdot L) = (g_0 + g_1 A)A(g_0 + g_1 A)^{-1}$ is contained in the conjugacy class of $f'(L)$ in $\mathfrak{g}$. On the other hand, choosing $g = g_0 \in G$, every element in the conjugacy class of $f'(L)$ is of the form $f'(g \cdot L)$.
This means $G(R[\![z]\!])$-orbits in $L \in (\mathrm{L}^{<0}G \cdot L_0) \cap \mathscr{G}_G^{(1)}$ are mapped into conjugacy classes of $\mathfrak{g}$. In particular there is a $G(R[\![z]\!])$-orbit inside $(\mathrm{L}^{<0}G \cdot L_0) \cap \mathscr{G}_G^{(1)}$ that is mapped to the minimal orbit of $\mathfrak{g}$. As both sides have a unique minimal non-trivial orbit above 0 ($\mathscr{G}_G$ since G has a connected Dynkin-Diagram connected and $\mathfrak{g}$ by 8.1), this concludes the proof. □

9 The Case of PGL_2

Let $G = \mathrm{PGL}_2$. Then the root system is A_1 and there is only one fundamental co-weight $\check{\omega}$. The only simple co-root is $2\check{\omega}$. There are two minimal fundamental co-weights, 0 and $\check{\omega}$. The order of dominant co-weights is linear above each of the two and the only minimal degenerations (λ, μ) are $\lambda = p \cdot \check{\omega}$ and $\mu = (p+2) \cdot \check{\omega}$ for $p \in \mathbb{N}$.

9.1 The characteristic polynomial of z
Let $G = \mathrm{SL}_n$, $\lambda \in \Lambda^+_{\mathrm{SL}_n}$ and $L \in \overline{\mathscr{G}_\lambda}(R)$. Let $\bar{L}$ be its image in $\mathrm{Grass}^z(nN, 2nN)(R)$. Then the characteristic polynomial $p(T)$ of z as an endomorphism of the R-module $\bar{L}$ is T^{nN}:
Let $A \in \mathrm{M}_{nN,2nN}(R)$ represent $\bar{L}$ as in 3.12. Let $J = \{j_1, \ldots, j_{nN}\} \subset \{1, \ldots, 2nN\}$ such that $\bar{L} \in U_J$. Then we can choose A such that the submatrix consisting of the rows with index in J is the identity matrix 1_{nN}. We know how z operates on $\big(R[z]/(z^{2N})\big)^n$: on the basis b_k (see 3.7) we have $z \cdot b_k = b_{k+2N}$ if $k + 2N \leq 2Nn$ and $z \cdot b_k = 0$ otherwise.
Let Z be the matrix representing z as a R-linear endomorphism of $\big(R[z]/(z^{2N})\big)^n$ with respect to the basis b_j. To calculate $p(T)$ we have to find a matrix $B \in M_{nN}(R)$ with

$$Z \cdot A = A \cdot B$$

and calculate the characteristic polynomial of B.
But using the special form of A we get that the j_i-th row on the right hand side is the i-th row of B. But the j_i-th row on the left hand side is the $(j_i - N)$-th row of A (if $j_i - N > 0$ and 0 otherwise).
This applies to all elements of U_J. As a result one sees that the columns of B are just some columns of A. Therefore the coefficients of $p(T)$ are given as polynomials on U_J. Choosing another J obviously gives a change of base of A and B. This does not change the characteristic polynomial of B and we see that the coefficients of $p(T)$ are regular functions on $\mathrm{Grass}^z(nN, 2nN)$.
This means we can define a closed subscheme V of $\overline{\mathscr{G}_\lambda}$ by the condition $p(T) = T^{nN}$.
But since z is nilpotent we know that all k-valued points of $\overline{\mathscr{G}_\lambda}$ lie in V. Because $\overline{\mathscr{G}_\lambda}$ is equipped with the reduced scheme-structure the assertion follows.

Theorem 9.2
Let $G = \mathrm{PGL}_2$ and λ and μ as above. Then

$$\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu} \cong \operatorname{Spec} k[T_1, T_2, T_3]/(T_1^{p+2} - T_2 T_3)$$

Proof. If p is even then $0 \leq \lambda < \mu$ and $\overline{\mathscr{G}_\mu} \subset Y_0$. We can take $\overline{\mathscr{G}_\mu} \subset \mathscr{G}_{\mathrm{SL}_n}$ using 5.6, as $\overline{\mathscr{G}_\mu}$ is equipped with the reduced structure.
Let $\breve{\alpha}$ be the simple co-root of SL_2 and let $m = 1/2p$. Under the above isomorphism L_λ is sent to $L_{m\breve{\alpha}}$ and L_μ is sent to $L_{(m+1)\breve{\alpha}}$. These are the special lattices generated by the matices $z^{m\breve{\alpha}} = \mathrm{diag}(z^m, z^{-m})$ and $z^{(m+1)\breve{\alpha}} = \mathrm{diag}(z^{m+1}, z^{-m-1})$ respectively.
Now we can use 5.1 and the construction in the proof of Lemma 3.6.
For SL_2 we have $\overline{\mathscr{G}_\mu} = \mathscr{G}^{(m+1)}_{\mathrm{SL}_2}$, since the order of dominant co-weights is linear in this case. This means we can identify $\overline{\mathscr{G}_\mu}$ with $\mathrm{Grass}^z(m+1, 2m+2)$ with respect to the basis $(z^{-m-1}e_1, z^{-m}e_1, \ldots, z^m e_1, z^{-m-1}e_2, z^{-m}e_2, \ldots, z^m e_2)$.

If p is odd then $\breve{\omega} \leq \lambda < \mu$ and $\overline{\mathscr{G}_\mu} \subset Y_1$. Let $m = 1/2(p+1)$. Then L_λ is represented by the matrix $\mathrm{diag}(z^{m+1}, z^{-m})$ and L_μ is represented by the matrix $\mathrm{diag}(z^{m+2}, z^{-m-1})$.

There is an embedding $f\colon \overline{\mathscr{G}_\mu} \cong \mathscr{G}_{\mathrm{SL}_n}$, given by multiplication with $\check{\omega}(z)$ using 5.6. Under this isomorphism $\mathscr{G}_{\mathrm{SL}_2}^{(m+1)}$, parametrizing lattices L such that

$$z^{m+1}k[\![z]\!]^2 \subseteq L \subseteq z^{-m-1}k[\![z]\!]^2$$

is sent to the subscheme of $\mathscr{G}_{\mathrm{PGL}}$ parametrizing L such that

$$z^{m+2}k[\![z]\!]^2 \subseteq L \subseteq z^{-m-1}k[\![z]\!]^2$$

This is clearly $\overline{\mathscr{G}_\mu}$, since the partial order of co-weights is linear above $\check{\omega}$.
We identify it in the usual way with $\mathrm{Grass}^z(4m+6, 2m+3)$ with respect to the basis $(z^{-m-1}e_1, z^{-m}e_1, \dots, z^m e_1, z^{m+1}e_1, z^{-m-1}e_2, z^{-m}e_2, \dots, z^m e_2, z^{m+1}e_2)$.

Let $r = 2m+2$ in the case of p being even or $r = 2m+4$ in the case of p being odd.
Since $\overline{\mathscr{G}_\mu} \cap \mathrm{L}^{<0}G \cdot z^\lambda \subset \mathcal{U}_\lambda^{(r)}$ by 3.13, $L \in (\overline{\mathscr{G}_\mu} \cap \mathrm{L}^{<0}G \cdot z^\lambda)(k)$ has to be of the form

$$L \hat{=} \left(\begin{array}{ccccc}
a_{1,1} & a_{1,2} & \dots\dots & & a_{1,r} \\
a_{2,1} & \ddots & & & a_{2,r} \\
\vdots & & \ddots & & \vdots \\
\vdots & & & \ddots & \vdots \\
a_{r-1,1} & \dots\dots & \dots\dots & \dots & a_{r-1,r} \\
1 & 0 & \dots\dots & & 0 \\
\hline
c_1 & c_2 & \dots\dots & & c_r \\
0 & 1 & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & 0 & 1 & 0 \\
0 & \dots\dots & & 0 & 1
\end{array}\right)$$

with respect to the basis $\left(z^{-m}e_1, z^{-m+1}e_1, \dots, z^{m-1}e_1, z^{-m}e_2, z^{-m+1}e_2, \dots, z^{m-1}e_2\right)$ where the size of each block is r. The line marks the point where the e_2-entries start.
By Lemma 5.13

$$L \in \mathrm{L}^{<0}G \cdot z^\lambda \Leftrightarrow \lim_{s\to 0} s^{-1} \cdot L = z^\lambda$$

We can apply this criterion to the above matrix and get

$$\lim_{s\to 0} s^{-1} \cdot L = \lim_{s\to 0} \left(\begin{array}{ccccc}
s^m a_{1,1} & s^m a_{1,2} & \dots\dots\dots & & s^m a_{1,r} \\
s^{m-1} a_{2,1} & s^{m-1} a_{2,2} & & & s^{m-1} a_{2,r} \\
\vdots & & \ddots & & \vdots \\
s^{-m+2} a_{r-1,1} & s^{-m+2} a_{r,2} & \cdots & s^{-m+2} a_{r-1,r-1} & s^{-m+2} a_{r-1,r} \\
s^{-m+1} & 0 & \dots\dots\dots & & 0 \\
\hline
s^m c_1 & s^m c_2 & \dots\dots\dots & & s^m c_r \\
0 & s^{m-1} & 0 & \dots\dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots\dots & 0 & s^{-m+2} & 0 \\
0 & \dots\dots & \dots & 0 & s^{-m+1}
\end{array}\right)$$

$$\equiv \lim_{s\to 0}\left(\begin{array}{ccccc} s^{2m-1}a_{1,1} & sa_{1,2} & s^2a_{1,3} & \cdots\cdots & s^{2m-1}a_{1,r} \\ s^{2m-2}a_{2,1} & a_{2,2} & sa_{2,3} & \cdots & s^{2m-2}a_{2,r} \\ \vdots & \vdots & \ddots & & \vdots \\ sa_{r-1,1} & s^{-2m+3}a_{r-1,1} & \cdots\cdots & a_{r-1,r-1} & sa_{r-1,r} \\ 1 & 0 & \cdots\cdots & \cdots\cdots & 0 \\ \hline s^{2m-1}c_1 & sc_2 & \cdots\cdots & \cdots\cdots & s^{2m-1}c_r \\ 0 & 1 & 0 & \cdots\cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots\cdots & 0 & 1 & 0 \\ 0 & \cdots\cdots & \cdots\cdots & 0 & 1 \end{array}\right)$$

as points in the Grassmannian.
For this to correspond to the point L_λ all entries under the diagonal of the upper half matrix must vanish except for the first column, i.e. $a_{i,j} = 0$ for $i \geq j$ and $2 \geq i \geq r-1$.
Using the z-stability yields

$$z \cdot L = \left(\begin{array}{ccccc} 0 & \cdots & \cdots & \cdots & 0 \\ a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,r} \\ a_{2,1} & 0 & a_{2,3} & \cdots & a_{2,r} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{r-1,1} & 0 & \cdots & 0 & a_{r-1,r} \\ \hline 0 & \cdots & \cdots & \cdots & 0 \\ c_1 & c_2 & \cdots & \cdots & c_r \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,r} \\ a_{2,1} & 0 & a_{2,3} & \cdots & a_{2,r} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{r-1,1} & 0 & \cdots & 0 & a_{r-1,r} \\ 1 & 0 & \cdots & \cdots & 0 \\ \hline c_1 & c_2 & \cdots & \cdots & c_r \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{array}\right) \cdot A$$

For suitable $A \in \mathrm{GL}_r(k)$.
It is clear that

$$A = \begin{pmatrix} a_{r-1,1} & 0 & \cdots & 0 & a_{r-1,r} \\ c_1 & \cdots & \cdots & \cdots & c_r \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

We obtain the following equations:

$$(a_{1,1},\dots,a_{1,r})\cdot A=(0,\dots,0) \tag{9}$$

$$(a_{i,1},\dots,a_{i,r})\cdot A=(a_{i-1,1},\dots,a_{i-1,r}) \text{ for } i=2,\dots,r-1 \tag{10}$$

$$(c_1,\dots,c_r)\cdot A=(0,\dots,0) \tag{11}$$

Using (10) all $a_{i,j}$ are given recursively by $a_{r-1,1} =: c_0$ and $a_{r-1,r} := d$ because all other entries in the $(r-1)$th row are zero. It can be seen easily that the zero entries in the other rows impose no further restriction.
Looking at (11) in more detail produces the following equations:

$$\begin{aligned} c_0c_1+c_1c_2&=0\\ (c_3,\dots,c_r)&=-c_2(c_2,\dots,c_{r-1}) \quad \text{so}\\ c_i&=c_2(-c_2)^{i-2} \quad \text{for } i=3,\cdots,r\\ dc_1+c_2c_r=0 &\Leftrightarrow dc_1+c_2^r=0 \end{aligned} \tag{12}$$

So we are left with the four variables c_1, c_2, c_0 and d.
Putting (9) and (10) together we get

$$(c_0,0,\dots,0,d)\cdot A^{r-1}=(0,\dots,0)$$

so we have to compute A^{r-1}. Luckily we are only interested in the first and last row of A^{r-1} and it is an easy calculation to see that those are

$$\begin{array}{ll} (c_0^{r-1}d, c_0d, c_0^2d,\dots,c_0^{r-2}d) & \text{and}\\ (c_1,c_2,-c_2^2,c_2^3,\dots c_2^{r-1}) & \text{respectively.} \end{array}$$

Thus the above equation yields

$$c_0^r+c_1d=0 \tag{13}$$

$$c_0^id+(-1)^{i+1}c_2^id=0 \qquad \text{for } i=1,\dots,r-1 \tag{14}$$

But looking at the trace of the matrix A we see $c_0 = c_2$ because $A = z|L$ has the characteristic polynomial T^r by 9.1. So the only condition we are left with is (13). This concludes the proof. □

10 Classification of minimal degenerations of $\mathrm{L}^{\geq 0}G$-orbits

We cite the following theorem from [Stembridge, 1998] which classifies all minimal degenerations.
Let $\lambda, \mu \in \Lambda_G^+$, $\lambda \leq \mu$. Denote by $I_{\lambda,\mu} := \mathrm{supp}(\mu-\lambda)$ the Dynkin subdiagram involving all simple co-roots appearing in the decomposition of $\mu-\lambda$ as a sum of simple co-roots. This is well defined since the simple co-roots form a basis.

It is obvious that if $(\lambda, \mu$ is a minimal degeneration, then $I_{\lambda,\mu}$ is connected.
Let $M_{\lambda,\mu}$ be the commutator of the Levi subgroup of G corresponding to $I_{\lambda,\mu}$ ($M_{\lambda,\mu}$ is uniquely determined by our choice of B and T as in section 7). Let $Q_{\lambda,\mu} \subset Q$ be the co-root lattice of $M_{\lambda,\mu}$ and let $\bar{\alpha}_{\lambda,\mu}$ be the short dominant co-root in the root-system of $M_{\lambda,\mu}$.
$Q_{\lambda,\mu}$ is generated by $\check{\alpha}_i$, $i \in I_{\lambda,\mu}$, In particular $\bar{\alpha}_{\lambda,\mu} \in \{\check{\alpha}_i \mid i \in I_{\lambda,\mu}\}$.
Let $J_{\lambda,\mu}$ be the Dynkin subdiagram of $I_{\lambda,\mu}$ consisting of the simple co-roots $\check{\alpha}_i$ for which $\langle \lambda, \alpha_i \rangle = 0$.
We denote by $C_{\min}(\lambda, \mu)$ the minimal conjugacy class in $\mathrm{Lie}(M_{\lambda,\mu})$ (see 8.1).

Theorem 10.1 ([Stembridge, 1998, Theorem 2.8])
A pair $\lambda, \mu \in \Lambda_G^+$ is a minimal degeneration if and only if one of the following conditions hold:

(a) $\mu - \lambda$ is a simple co-root $\check{\alpha}_i, i \in I_G$

(b) $I_{\lambda,\mu} = J_{\lambda,\mu}$ and $\mu - \lambda = \bar{\alpha}_{\lambda,\mu}$

(c) $I_{\lambda,\mu} = J_{\lambda,\mu} \cup \{i\}$, $I_{\lambda,\mu}$ is of type C, α_i is short, $\langle \lambda, \check{\alpha}_i \rangle = 1$, and $\mu - \lambda = \bar{\alpha}_{\lambda,\mu}$

(d) $I_{\lambda,\mu} = J_{\lambda,\mu} \cup \{i\}$, $I_{\lambda,\mu} \cong G_2$, α_i is short, $\langle \lambda, \check{\alpha}_i \rangle \in \{1, 2\}$, and $\mu - \lambda \in Q_{\lambda,\mu}$ is the sum of two simple roots that generate G_2

Using our explicit calculations from sections 8 and 9 we can describe the scheme $(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu}$ in the cases a) and b) of the theorem above as follows:

Theorem 10.2
Let (λ, μ) be a minimal degeneration of coweights.

(a) If $\mu - \lambda$ is a simple co-root $\check{\alpha}_i, i \in I_G$ then

$$(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu} \cong \operatorname{Spec} k[T_1, T_2, T_3]/(T_1^{\lambda_i+2} - T_2 T_3)$$

(b) If $I_{\lambda,\mu} = J_{\lambda,\mu}$ and $\mu - \lambda = \bar{\alpha}_{\lambda,\mu}$ then

$$(\mathrm{L}^{<0}G \cdot L_\lambda) \cap \overline{\mathscr{G}_\mu} \xrightarrow{\cong} \overline{C_{\min}(\lambda, \mu)}$$

Proof. Using Lemma 7.6 we can restrict ourselves to the Levi subgroup $M_{\lambda,\mu}$ corresponding to the Dynkin diagram $I_{\lambda,\mu}$ and the dominant co-weights λ_M and μ_M. In the case a) we simply have $I_{\lambda,\mu} = A_1$, corresponding to the case of PGL_2. In the case b) $\langle \lambda, \check{\alpha}_i \rangle = 0$ for all $i \in I_{\lambda,\mu}$, thus the restriction

$$\mu_{I_{\lambda,\mu}} := \sum_{i \in I_{\lambda,\mu}} \langle \mu, \check{\alpha}_i \rangle \cdot \check{\omega}_i$$

of μ to $I_{\lambda,\mu}$ is $\bar{\alpha}_{\lambda,\mu}$. This means we are in the situation of Lemma 8.3. $\square$

10.3
A natural question arising from theorem 10.2 is: Given a reductive group G, which singularities can arise as minimal degeneration singularities in $\mathscr{G}_G$. This question can be answered in terms of connected subdiagrams $I \subset I_G$. Denote by $\bar{\alpha}_I$ the short dominant co-root of I.

(a) For each fundamental co-weight $\check{\omega}$ of G and each $p \in \mathbb{N}$ we can choose $\mu = (p+2)\check{\alpha}$ to get a Kleinian singularity of the type in a), namely a_{p+1}. By the proof of Theorem 10.2 we know that these are the only singularities occurring in case a).

(b) By Theorem 10.2 the singularity in b) is uniquely determined by I and obviously each connected subdiagram of I_G gives rise to a singularity of this type by setting $\mu = \bar{\alpha}_I$ and $\lambda = 0$. Thus for each such I we get a Kleinian singularity of type I.

(c) If I_G contains a diagram of type C the case c) of Theorem 10.1 can occur but the situation here seems to be more involved. Let I be a subdiagram of I_G of type C_p. Let (λ, μ) be a minimal degeneration with $I = I_{\mu,\lambda}$. Using Lemma 7.6 with M corresponding to the Dynkin diagram I (as in the proof of Theorem 10.2) we know that the singularity for this minimal degeneration depends only on μ_M and λ_M. But the conditions in case c) require λ_M to be the unique long co-root $\check{\alpha}_i$ with $i \in I$ and μ_M to be $\bar{\alpha}_I - \lambda_M$. This means the singularity is uniquely determined by I and thus by p. It is called ac_p. We note that this singularity does not arise for every $I \subset I_G$ of type C_p, see 10.4.

(d) If I_G contains a diagram of type G_2 we can argue as in c). Given $I \subset I_G$ of type G_2 there are two choices for λ_M: $\lambda_M = \check{\alpha}_i$ with $\check{\alpha}_i$ the unique long co-root with $i \in I$ or $\lambda_M = 2\check{\alpha}_i$. This determines μ_M. Thus I together with one of the two choices determine the singularity. They are called ag_2 and cg_2 respectively.

Summarizing the above we see that for each connected subdiagram I there is a Kleinian singularity of the according type, if I is of type C_p there is another singularity ac_p and if I is of type G_2 then there are two more singularities arising from I, ag_2 and cg_2.

10.4
To illustrate the different cases of theorem 10.1 we observe that case c) is for example not possible for $G = \mathrm{Sp}_{2g}$ even though it is of type C: Assume we are given $\mu, \lambda \in \Lambda_{\mathrm{Sp}_{2g}}$ as in c). It is obvious that the short simple co-root of I is $\check{\alpha}_g$, the short simple co-root of $I_{\mathrm{Sp}_{2g}}$ (this is the only possibility for a subdiagram of C to be of type C again). If $\lambda = \sum_{i \in I_G} \lambda_i \check{\omega}_i$ then $\langle \lambda, \check{\alpha}_g \rangle = \lambda_g$.
But $\check{\omega}_g = (1/2, \ldots, 1/2)$, so λ_g is even and therefore $\neq 1$.

Now we briefly discuss the dimension of the Kleinian singularities and the singularities ac_p.
As a result of Lemma 5.3 c) and Lemma 6.7 a) iii) we know that $\dim\left(\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}\right)$ only depends on $\mu - \lambda$.
Let $I \subset G_I$ of type C_p. Then given μ and λ as in a or b) with $I = I_{\lambda,\mu}$ we get the same $\mu - \lambda$ as with μ and λ as in c).
This means that $\dim(c_n) = \dim(ac_n)$.
Below we give a list of the dimensions of the singularities corresponding to the possible choices of I in the classical cases.

- $I = A_l$: $2\rho = \sum_{i=1}^{l+1}(l-2(i-1))e_i$, $\bar{\alpha} = \sum_{i\in I}\alpha_i = (1,0,\dots,0,-1) \Rightarrow \langle 2\rho, \bar{\alpha}\rangle = 2l$
- $I = B_l$: $2\rho = \sum_{i=1}^{l}(2(l-i)+1)e_i$, $\bar{\alpha} = (1,1,0,\dots,0) \Rightarrow \langle 2\rho,\bar{\alpha}\rangle = 4(l-1)$
- $I = C_l$: $2\rho = \sum_{i=1}^{l} 2(l-i+1)e_i$, $\bar{\alpha} = (1,0,\dots,0) \Rightarrow \langle 2\rho,\bar{\alpha}\rangle = 2l$
- $I = D_l$: $2\rho = \sum_{i=1}^{l-1} 2(l-i)e_i$, $\bar{\alpha} = (1,1,0,\dots,0) \Rightarrow \langle 2\rho,\bar{\alpha}\rangle = 4l-6$
- $I = G_2$: $2\rho = 2(-1,-2,3)$, $\bar{\alpha} = 1/3(-2,-2,4) \Rightarrow \langle 2\rho,\bar{\alpha}\rangle = 12$

Since for every singularity of the type in case c) there is a singularity of the same dimension of the type in case b) one can ask whether those are the same. The answer to this question is negative according to [Juteau, 2008], see the introduction.
Below we give an example for the case ac_2. Unfortunately from the calculation it is not clear that this singularity is not the Kleinian singularity c_2.

11 Calculation of the Singularity of type ac_2

An example for a minimal degeneration singularity of type ac_2 is given by the co-weights $\lambda = (1/2, 1/2)$ and $\mu = (3/2, 1/2)$ in the root system C_2.
There are two minimal co-weights in $\Lambda^+_{\mathrm{PSp}_4}$, 0 and $\lambda = (1/2,1/2)$, giving two connected components of $\mathscr{G}_{\mathrm{PSp}_4}$. The connected component Y_0 belonging to the co-weight 0 is naturally isomorphic to $\mathscr{G}_{\mathrm{Sp}_4}$, analogous to the situation of PGL (see 5.6).
The connected component Y_1 corresponding to λ is characterized by the condition that $L \in Y_1$ if and only if L is represented by a matrix $g \in \mathrm{GSp}_4(k(\!(z)\!))$ such that $g^t \cdot J \cdot g = z^{-1}J$ where J is the symplectic form given by the matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(This again is analogous to the situation for PGL. In general the connected components Y_i are characterized by the condition $g^t \cdot J \cdot g = z^{-1}J$ for all $g \in \mathrm{PSp}$. The proof is the same as for PGL.)
Then L_λ is represented by the matrix $z^\lambda := \mathrm{diag}(1,1,z^{-1},z^{-1})$ and L_μ is represented by the matrix $z^\mu := \mathrm{diag}(z,1,z^{-1},z^{-2})$.

The inclusion $\mathrm{Sp}_4 \subset \mathrm{GSp}_4$ induces a closed immersion $\mathscr{G}_{\mathrm{Sp}_4} \hookrightarrow \mathscr{G}_{\mathrm{GSp}_4}$. Multiplication with z^λ induces an isomorphism of $\mathscr{G}_{\mathrm{GSp}_4}$ that maps $\mathscr{G}_{\mathrm{Sp}_4}$ isomorphically to the closed sub-k-space defined by the condition $g^t \cdot J \cdot g = z^{-1}J$, which is obviously isomorphic to Y_1. Using this fact we can calculate inside $\mathscr{G}_{\mathrm{GSp}_4}$.
To compute $\mathrm{L}^{<0}G \cdot L_\lambda \cap \overline{\mathscr{G}_\mu}$ we observe that $\mathscr{G}_\mu \subset z^\lambda \cdot \mathscr{G}^{(1)}_{\mathrm{Sp}_4} \subset \mathscr{G}_{\mathrm{GSp}_4}$ since $L_{(1,0)} \in \mathscr{G}^{(1)}_{\mathrm{Sp}_4}$ and $z^\lambda \cdot L_{(1,0)} = L_\mu$ (where $L_{(1,0)}$ is the lattice belonging to the coweight $(1,0) \in \Lambda^+_{\mathrm{Sp}_4}$, $L_(1,0)$ being represented by the matrix $\mathrm{diag}(z,1,1,z^{-1})$). The closed subscheme $z^\lambda \cdot \mathscr{G}^{(1)}_{\mathrm{Sp}_4}$ contains the lattices $L \in \mathscr{G}_{\mathrm{GSp}_4}$, such that

$$z^1 R[\![z]\!]^4 \subset L \subset z^{-2} R[\![z]\!]^4$$

Analogously to Lemma 3.6 we obtain a closed immersion

$$\lambda(z) \cdot \mathscr{G}^{(1)}_{\mathrm{Sp}_4} \hookrightarrow \mathrm{Grass}^z_{\mathrm{GSp}4}(6,12)$$

sending a lattice L to its quotient in $z^{-2}R[\![z]\!]^4/z \cdot R[\![z]\!]^4$. Consider the basis

$$(e_1, \ldots, e_4, z^{-1}e_1, \ldots, z^{-1}e_4, z^{-2}e_1, \ldots, z^{-2}e_1)$$

of $z^{-2}R[\![z]\!]^4/z \cdot R[\![z]\!]^4$, where e_i is the standard basis of $R[\![z]\!]^4$.
We write U_λ for the image of $\mathcal{U}_\lambda \cap z^\lambda \cdot \mathscr{G}^{(1)}_{\mathrm{Sp}_4}$ in the Grassmannian analog to the description in 3.13. Then $L \in U_\lambda$ is of the form

$$L = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1_2 & 0 \\ A_1 & A_2 & A_3 \\ 0 & 0 & 1_2 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$$

written as a block matrix with all entries being 2×2-matrices. Using the criterion of Lemma 5.13 on L we get:

$$\lim_{s \to 0} s^{-1} L = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1_2 & 0 \\ s \cdot A_1 & s \cdot A_2 & s \cdot A_3 \\ 0 & 0 & s \cdot 1_2 \\ s^2 \cdot A_4 & s^2 \cdot A_5 & s^2 \cdot A_6 \\ s^2 \cdot A_7 & s^2 \cdot A_8 & s^2 \cdot A_9 \end{pmatrix}$$

Multiplying from the right by $\left(\begin{smallmatrix} 1_2 & 0 & 0 \\ 0 & 1_2 & 0 \\ 0 & 0 & s^{-1} \cdot 1_2 \end{smallmatrix}\right)$ yields $A_3 = 0$. Observe that z acts on the lattice corresponding to L as multiplication by

$$\begin{pmatrix} A_1 & A_2 & 0 \\ 0 & 0 & 1_2 \\ A_7 & A_8 & A_9 \end{pmatrix}$$

from the right. So z-stability of the lattice yields the equations

$$A_1^2 = A_4 \tag{15}$$
$$A_1 A_2 = A_5 \tag{16}$$
$$A_2 = A_6 \tag{17}$$
$$A_8 + A_9^2 = 0 \tag{18}$$

and

$$A_4A_1 + A_6A_7 = 0 \tag{19}$$
$$A_4A_2 + A_6A_8 = 0 \tag{20}$$
$$A_5 + A_6A_9 = 0 \tag{21}$$
$$A_7A_1 + A_9A_7 = 0 \tag{22}$$
$$A_7A_2 + A_9A_8 = 0 \tag{23}$$

Using (15)-(18) we only have to deal with A_1, A_2, A_7 and A_9 so L is of the form

$$L = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & 1_2 & 0 \\ A_1 & A_2 & 0 \\ 0 & 0 & 1_2 \\ A_1^2 & A_1A_2 & A_2 \\ A_7 & -A_9^2 & A_9 \end{pmatrix}$$

We rewrite (19)-(23) using (15) - (18):

$$A_1^3 + A_2A_7 = 0 \tag{24}$$
$$A_1^2A_2 - A_2A_9^2 = 0 \tag{25}$$
$$A_1A_2 + A_2A_9 = 0 \tag{26}$$
$$A_7A_1 + A_9A_7 = 0 \tag{27}$$
$$A_7A_2 - A_9^3 = 0 \tag{28}$$

We know that $\overline{\mathscr{G}_\mu}$ consists of the orbits of all dominant co-weights $\nu \leq \mu$. But the Grassmannian considered here also contains the orbit belonging to the co-weight $(1, 1, -2, -2)$. To exclude this orbit we add the condition that the submatrix corresponding to the z^{-2} entries has rank one, i.e.

$$\operatorname{rank} \begin{pmatrix} A_1^2 & A_1A_2 & A_2 \\ A_7 & -A_9^2 & A_9 \end{pmatrix} \leq 1 \tag{29}$$

To see that this condition excludes all orbits in $\mathrm{Grass}^z_{\mathrm{GSp4}}(6, 12)$ but the orbits $\mathscr{G}_\nu$ with $\nu \leq \mu$ we give a full list of the orbits in $\mathrm{Grass}^z_{\mathrm{GSp4}}(6, 12)$ by listing the corresponding fundamental co-weights. Let $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$. There are the following conditions:

- $\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4$ since ν dominant
- $\nu_1 + \nu_3 = -1$ and $\nu_2 + \nu_4 = -1$ as only those L_ν are in the component Y_1 we are interested in.
- $-2 \leq \nu_1, \nu_2, \nu_3, \nu_4$ since all lattices in $\mathrm{Grass}^z_{\mathrm{GSp4}}(6, 12)$ are required to be contained in $z^{-2}k[\![z]\!]^4$ by construction.

This leaves only $\nu = (0, 0, -1, -1) = \lambda$, $\nu = (1, 0, -1, -2) = \mu$ and $\nu = (1, 1, -2, -2)$. The last one is the only one with $\nu \not\leq \mu$ and its orbit is obviously excluded by condition (29).

The final condition that we have to impose on L is to belong to a lattice actually generated by an element of GSp_4. This means

$$z \cdot L_0^t \cdot J \cdot L_0 \in \mathrm{M}_n(k[\![z]\!]) \tag{30}$$

where L_0 is the matrix associated to the given representation of L:

$$L_0 = \begin{pmatrix} 1_2 + z^{-1}A_1 + z^{-2}A_1^2 & z^{-2}A_2 \\ z^{-2}A_7 & z^{-1}1_2 + z^{-2}A_9 \end{pmatrix}$$

Using the special form of L it follows from an easy calculation that this condition is equivalent to $z \cdot L_0^t \cdot J \cdot L_0 = J$.

In the following we write $\bar{1}_2$ for $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\bar{A}$ for $\bar{1}_2 \cdot A$ where A is a (2×2) matrix. Carrying out the multiplication in (30) and sorting by z-exponents we derive

$$\bar{A}_2^t = A_2^t \bar{1}_2 \tag{31}$$
$$\bar{A}_7 = A_7^t \bar{1}_2 \tag{32}$$
$$\bar{A}_1 = -A_9^t \bar{1}_2 \tag{33}$$
$$A_1^t \bar{A}_7 = A_7^t \bar{A}_1 \tag{34}$$
$$(A_1^t)^2 \bar{A}_7 = A_7^t \bar{A}_1^2 \tag{35}$$
$$A_2^t \bar{A}_7 = A_9^t \bar{A}_1^2 \tag{36}$$
$$\bar{A}_1{}^2 = A_9^t \bar{A}_1 \tag{37}$$
$$A_2^t \bar{A}_9 = A_9^t \bar{A}_2 \tag{38}$$

From now on we write

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

By (33) A_9 is determined by A_1. Also $\bar{A}_2$ and $\bar{A}_7$ are symmetric by (31) and (32). More explicitly:

$$A_9 = \begin{pmatrix} -d_1 & -b_1 \\ -c_1 & -a_1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & a_2 \end{pmatrix} \qquad A_7 = \begin{pmatrix} a_7 & b_7 \\ c_7 & a_7 \end{pmatrix} \tag{39}$$

Now we want to simplify (29):

$$\begin{pmatrix} A_1^2 & A_1A_2 & A_2 \\ A_7 & -A_9^2 & A_9 \end{pmatrix} \tag{40}$$

$$= \begin{pmatrix} a_1^2 + b_1c_1 & b_1(a_1+d_1) & a_1a_2 + b_1c_2 & a_1b_2 + a_2b_1 & a_2 & b_2 \\ c_1(a_1+d_1) & d_1^2 + b_1c_1 & a_2c_1 + d_1c_2 & c_1b_2 + d_1a_2 & c_2 & a_2 \\ a_7 & b_7 & -d_1^2 - b_1c_1 & -b_1(a_1+d_1) & -d_1 & -b_1 \\ c_7 & a_7 & -c_1(a_1+d_1) & -a_1^2 - c_1b_1 & -c_1 & -a_1 \end{pmatrix} \tag{41}$$

Looking at the (2×2) minors of $\left(\begin{smallmatrix} A_2 \\ A_9 \end{smallmatrix}\right)$ we get

$$b_1c_2 = a_2d_1 \qquad a_2b_1 = b_2d_1 \tag{42}$$
$$a_2c_1 = c_2a_1 \qquad c_1b_2 = a_2a_1. \tag{43}$$

Using this we see that $A_1A_2 = (a_1 + d_1)A_2$. Since $\operatorname{rank} A_1 \leq 1$ it follows that $A_1^2 = (a_1 + d_1)A_1$, thus the third and fourth column in the above matrix are multiples of the last two columns. This means we can write (29) in the following way:

$$\operatorname{rank}\begin{pmatrix} a_1(a_1+d_1) & b_1(a_1+d_1) & a_2 & b_2 \\ c_1(a_1+d_1) & d_1(a_1+d_1) & c_2 & a_2 \\ a_7 & b_7 & -d_1 & -b_1 \\ c_7 & a_7 & -c_1 & -a_1 \end{pmatrix} \leq 1 \tag{44}$$

We claim that all the conditions we have not used yet, i.e. (24)-(28) and (34)-(38) follow from (44) and (39). Indeed, (24) and (28) are equivalent and (26) implies (25).
Using the (2×2) minors of (44) the other equations follow by an easy calculation. Therefore the only condition left is (44).

Since it is not clear from the equation (44) what kind of singularity this is, we used a computer to verify some properties. We confirmed that the dimension is 4, as follows from the discussion at the end of section 10. Also we verified that there is an isolated singularity at the origin. Attempts to validate that the singularity is normal or Cohen-Macaulay failed.

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